Modern tools for Collider predictions in QCD: Calcutta lectures

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Abstract

This is a rough draft of notes for lectures which I am giving in Calcutta, provided for the benefit of the students in Calcutta. The notes are deficient in many regards, not the least of which is the inadequate referencing. If the notes are prepared for wider dissemination, this will need to be rectified. Part of this material was prepared in collaboration with Kunszt, Melnikov and Zanderighi for a forthcoming review.

Contents

1 Gamma matrices and the Dirac Eqn.

1.1 Definition of γ matrices

Both representations satisfy the same commutation relations.

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{1.1}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.2}$$

$$\gamma_R \;= \frac{1}{2}(1+\gamma_5),\quad \gamma_L \;= \frac{1}{2}(1-\gamma_5) \tag{1.3}$$

1.2 Fierz transformation

$$\Lambda^{(i)}_{32} \otimes \Lambda^{(i)}_{14} = \sum_{j=1}^{5} \lambda_{ij}\, \Lambda^{(j)}_{12} \otimes \Lambda^{(j)}_{34}$$

where $\Lambda^{(i)} = (1, \gamma_\mu, \sigma_{\mu\nu}/\sqrt{2}, \gamma_\mu\gamma_5, \gamma_5)$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

$$\lambda_{ij} = \frac{1}{4}\begin{pmatrix} +1 & +1 & +1 & -1 & +1 \\ +4 & -2 & 0 & -2 & -4 \\ +6 & 0 & -2 & 0 & +6 \\ -4 & -2 & 0 & -2 & +4 \\ +1 & -1 & +1 & +1 & +1 \end{pmatrix}, \tag{1.4}$$

Using this relation it is easy to show that

$$(\gamma_\mu\gamma_L)_{32} \otimes (\gamma_\mu\gamma_L)_{14} = -(\gamma_\mu\gamma_L)_{12} \otimes (\gamma_\mu\gamma_L)_{34} \tag{1.5}$$

and that

$$(\gamma_\mu\gamma_R)_{32} \otimes (\gamma_\mu\gamma_L)_{14} = 2(\gamma_R)_{12} \otimes (\gamma_L)_{34} \tag{1.6}$$

1.3 Dirac eqn. Massless fermions

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

  $$\gamma^0 \;=\; \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i \;=\; \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 \;=\; \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{1.7}$$

  The Weyl representation is more suitable at high energy

  $$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i \;=\; \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 \;=\; \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \tag{1.8}$$

  In the Weyl representation upper and lower components have different helicities.

  $$\gamma_R \;= \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \gamma_L \;= \frac{1}{2}(1-\gamma_5) == \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{1.9}$$

- Both representations satisfy the same commutation relations.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{1.10}$$

- in the Weyl representation $\gamma^0 \gamma^i \;=\; \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.

In order to derive an explicit solution for the massless Dirac equation $\not{p}u(p) = 0$ it is useful to write out an explicit expression for $\not{p} = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3$ in the Weyl representation.

$$\not{p} = \begin{pmatrix} 0 & 0 & p^+ & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & p^- \\ p^- & -p^1 + ip^2 & 0 & 0 \\ -p^1 - ip^2 & p^+ & 0 & 0 \end{pmatrix} , \tag{1.11}$$

where $p^\pm \;=\; p^0 \pm p^3$.

- The massless spinors solns of Dirac eqn are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix}, \tag{1.12}$$

where

$$e^{\pm i\varphi_p} \;\equiv\; \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} \;=\; \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}} , \qquad p^\pm \;=\; p^0 \pm p^3. \tag{1.13}$$

Note that in the limit $p^+ = p^1 = p^2 = 0$ these solutions become

$$u_+(p) = \begin{bmatrix} 0 \\ \sqrt{2p^0} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2p^0} \\ 0 \end{bmatrix}, \tag{1.14}$$

In this representation the Dirac conjugate spinors are

$$\overline{u_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}\right] \tag{1.15}$$

$$\overline{u_-}(p) = \left[\sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right] \tag{1.16}$$

- Normalization

$$u_\pm^\dagger u_\pm = 2p^0 \tag{1.17}$$

Introduce a bra and ket notation spinors corresponding to (massless) momenta p_i, $i = 1, 2, \ldots, n$ labelled by the index i

$$|i^\pm\rangle \;\equiv\; |p_i^\pm\rangle \;\equiv\; u_\pm(p_i) \;=\; v_\mp(p_i), \tag{1.18}$$
$$\langle i^\pm| \equiv\; \langle p_i^\pm| \equiv\; \overline{u_\pm}(p_i) \;=\; \overline{v_\mp}(p_i). \tag{1.19}$$

We define the basic spinor products by

$$\langle ij\rangle \;\equiv\; \langle i^-|j^+\rangle \;=\; \overline{u_-}(p_i)u_+(p_j), \;\; [ij] \;\equiv\; \langle i^+|j^-\rangle \;=\; \overline{u_+}(p_i)u_-(p_j). \tag{1.20}$$

The helicity projection implies that products like $\langle i^+|j^+\rangle$ vanish.

$$\langle i+|j+\rangle = \langle i-|j-\rangle = \langle ii\rangle = [ii] = 0 \tag{1.21}$$

$$\langle ij\rangle = -\langle ji\rangle, \;\; [ij] = -[ji] \tag{1.22}$$

It is appropriate here to comment on alternative notations. Since the left-handed and right-handed spinors occupy different subspaces, we write them in terms in terms of two-index spinors, (also called holomorphic and anti-holomorphic spinors).

$$|j- >= (\lambda_j)_\alpha, \quad |j+ >= (\tilde{\lambda}_j)_{\dot{\alpha}} \tag{1.23}$$

In terms of these spinors we see that

$$\begin{aligned} \langle j\,k\rangle &= \varepsilon^{\alpha\beta}(\lambda_j)_\alpha(\lambda_k)_{\dot{\beta}} \\ [j\,k] &= \varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_j)_{\dot{\alpha}}(\tilde{\lambda}_k)_{\dot{\beta}} \end{aligned} \tag{1.24}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor in two dimensions. The two different types of indices, dotted and undotted, both run from 1 to 2. Dotted indices are only contracted with other dotted indices, and undotted indices are only contracted with other undotted indices.

1.4 Spinor products

We get explicit formulae for the spinor products valid for the case when both energies are positive, $p_i^0 > 0, \quad p_j^0 > 0$

$$\begin{aligned} \langle i\,j\rangle &= \sqrt{p_i^- p_j^+}e^{i\varphi_{p_i}} - \sqrt{p_i^+ p_j^-}e^{i\varphi_{p_j}} \;=\; \sqrt{|s_{ij}|}e^{i\phi_{ij}}, \\ [i\,j] &= \sqrt{p_i^+ p_j^-}e^{-i\varphi_{p_j}} - \sqrt{p_i^- p_j^+}e^{-i\varphi_{p_i}} \;=\; -\sqrt{|s_{ij}|}e^{-i\phi_{ij}} \end{aligned} \tag{1.25}$$

where $s_{ij} \;=\; (p_i+p_j)^2 \;=\; 2p_i\cdot p_j$, and

$$\cos\phi_{ij} \;=\; \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}|p_i^+ p_j^+}}\,, \qquad \sin\phi_{ij} \;=\; \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{\sqrt{|s_{ij}|p_i^+ p_j^+}}\,. \tag{1.26}$$

- The spinor products are, up to a phase, square roots of Lorentz products.
- *For real momenta* we have that $\langle i\,j\rangle^* \;=\; [j\,i]$, Note, however, that for complex momenta this is no longer true.
- The collinear limits of massless gauge amplitudes have square-root singularities; spinor products lead to very compact analytic representations of gauge amplitudes. By explicit construction we can show that

$$|B+\rangle\langle C-| - |C+\rangle\langle B-| = \langle C-|B+\rangle\gamma_R \tag{1.27}$$

$$\langle pq\rangle = \langle p-|q+\rangle, \quad [pq] = \langle p+|q-\rangle$$
$$\langle p\pm|\gamma_\mu|p\pm\rangle = 2p_\mu$$
$$\langle p+|q+\rangle = \langle p-|q-\rangle = \langle pp\rangle = [pp] = 0$$
$$\langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp]$$
$$2|p\pm\rangle\langle q\pm| = \tfrac{1}{2}(1\pm\gamma_5)\gamma^\mu\langle q\pm|\gamma_\mu|p\pm\rangle$$
$$\langle pq\rangle^* = -\mathrm{sign}(p\cdot q)[pq] = \mathrm{sign}(p\cdot q)[qp]$$
$$|\langle pq\rangle|^2 = \langle pq\rangle\langle pq\rangle^* = 2|p\cdot q| \equiv |s_{pq}|$$
$$\langle pq\rangle[qp] = 2p\cdot q \equiv s_{pq}$$
$$\langle p\pm|\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}|q\pm\rangle = \langle q\mp|\gamma_{\mu_{2n+1}}\ldots\gamma_{\mu_1}|p\mp\rangle$$
$$\langle p\pm|\gamma_{\mu_1}\ldots\gamma_{\mu_{2n}}|q\mp\rangle = -\langle q\pm|\gamma_{\mu_{2n}}\ldots\gamma_{\mu_1}|p\mp\rangle$$
$$\langle AB\rangle\langle CD\rangle = \langle AD\rangle\langle CB\rangle + \langle AC\rangle\langle BD\rangle, \quad \text{(Schouten)}$$
$$\langle A+|\gamma_\mu|B+\rangle\langle C-|\gamma^\mu|D-\rangle = 2[AD]\langle CB\rangle, \quad \text{(Fierz)}$$
$$\langle A\pm|\gamma^\mu|B\pm\rangle\ \gamma_\mu = 2\left[\ |A\mp\rangle\langle B\mp| + |B\pm\rangle\langle A\pm|\ \right], \quad \text{(Fierz + Chargeconjugation)}$$

Table 1.1: Relations for massless spinors

Thus we get the Schouten identity

$$\langle A-|B+\rangle\langle C-|D+\rangle - \langle A-|C+\rangle\langle B-|D+\rangle = \langle C-|B+\rangle\langle A-|D+\rangle$$

or written more concisely,

$$\begin{aligned} \langle A\,B\rangle\langle C\,D\rangle - \langle A\,C\rangle\langle B\,D\rangle &= \langle C\,B\rangle\langle A\,D\rangle \\ [A\,B][C\,D] - [A\,C][B\,D] &= [C\,B][A\,D] \end{aligned} \tag{1.28}$$

For polarization with momentum k and gauge vector b

$$\varepsilon^\pm_\mu(k,b) = \pm\frac{\langle k\pm|\gamma_\mu|b\pm\rangle}{\sqrt{2}\langle b\mp|k\pm\rangle} \tag{1.29}$$

Hence we have that

$$\varepsilon^+_\mu(k,b) = \frac{\langle k+|\gamma_\mu|b+\rangle}{\sqrt{2}\langle bk\rangle}, \quad \varepsilon^-_\mu(k,b) = \frac{\langle k-|\gamma_\mu|b-\rangle}{\sqrt{2}[kb]} \tag{1.30}$$

and

$$\gamma^\mu\varepsilon^+_\mu(k,b) = \frac{\sqrt{2}\Big[|k-\rangle\langle b-| + |b+\rangle\langle k+|\Big]}{\langle bk\rangle} \tag{1.31}$$

$$\gamma^\mu \varepsilon^-_\mu(k,b) = \frac{\sqrt{2}\Big[|k+\rangle\langle b+| + |b-\rangle\langle k-|\Big]}{[kb]} \tag{1.32}$$

Different choices of the vector b correspond to different choices of gauge. Thus

$$\begin{aligned}
\varepsilon^+_\mu(k,b) - \varepsilon^+_\mu(k,c) &= \frac{\langle k+|\gamma_\mu|b+\rangle}{\sqrt{2}\langle bk\rangle} - \frac{\langle k+|\gamma_\mu|c+\rangle}{\sqrt{2}\langle ck\rangle} \\
= \quad & \frac{1}{\sqrt{2}\langle bk\rangle\langle ck\rangle}\Big[\langle k+|\gamma_\mu|b+\rangle\langle ck\rangle - \langle k+|\gamma_\mu|c+\rangle\langle bk\rangle\Big] \\
= \quad & \frac{1}{\sqrt{2}\langle bk\rangle\langle ck\rangle}\Big[\langle k+|\gamma_\mu|k+\rangle\langle cb\rangle\Big] = \frac{\sqrt{2}\langle cb\rangle}{\langle bk\rangle\langle ck\rangle}k_\mu \qquad (1.33)
\end{aligned}$$

where we have used Eq. (1.27).

1.5 Charge conjugation

In QED we have

$$\Big((+i\nabla^\mu + eA^\mu)\gamma_\mu - m\Big)\psi = 0 \tag{1.34}$$

Taking the complex conjugate

$$\Big((-i\nabla^\mu + eA^\mu)\gamma^*_\mu - m\Big)\psi^* = 0 \tag{1.35}$$

The equation satisfied by the charge conjugate state is

$$\Big((+i\nabla^\mu - eA^\mu)\gamma_\mu - m\Big)\psi_c = 0 \tag{1.36}$$

The operation of charge conjugation is therefore given by

$$\psi_c = C\gamma^0\psi^* \tag{1.37}$$

where the matrix C is determined up to a phase by the condition $(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1} = -\gamma^\mu$.

Since for our representation $\gamma^0\gamma^{\mu\,*}\gamma^0 = \gamma^{\mu\,T}$ the defining condition on matrix C can be written

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T} \tag{1.38}$$

We choose the phase such that

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & \mathbf{0} \\ \mathbf{0} & -i\sigma^2 \end{pmatrix} \tag{1.39}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{1.40}$$

so that $C^T = C^{-1} = -C$ For free particle spinors we have that

$$v_\pm(p) = C\overline{u}^T_\pm(p), \tag{1.41}$$

$$v_+(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} 0 \\ m \\ p_x - ip_y \\ -p^+ \end{bmatrix} \tag{1.42}$$

$$v_-(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} p^+ \\ p_x + ip_y \\ -m \\ 0 \end{bmatrix} \tag{1.43}$$

In this representation the Dirac conjugate spinors are

$$\overline{v_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \frac{1}{\sqrt{p^+}}\left[p_x + ip_y, -p^+, 0, m\right] \tag{1.44}$$

$$\overline{v_-}(p) = \frac{1}{\sqrt{p^+}}\left[-m, 0, p^+, p_x - ip_y\right] \tag{1.45}$$

Note that

$$\sum_\lambda u_\lambda(p)\bar{u}_\lambda(p) = \not{p} + mI \tag{1.46}$$

$$\sum_\lambda v_\lambda(p)\bar{v}_\lambda(p) = \not{p} - mI \tag{1.47}$$

and that

$$\bar{u}_{\lambda_1}u_{\lambda_2} = +2m\delta_{\lambda_1\lambda_2} \tag{1.48}$$

$$\bar{v}_{\lambda_1}v_{\lambda_2} = -2m\delta_{\lambda_1\lambda_2} \tag{1.49}$$

and that

$$\bar{u}_{\lambda_1}(P)v_{\lambda_2}(P) = \bar{v}_{\lambda_1}(P)u_{\lambda_2}(P) = 0 \tag{1.50}$$

In summary

$$u_+(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} p^+ \\ p_x + ip_y \\ m \\ 0 \end{bmatrix}, \qquad \overline{u_+}(p) = \frac{1}{\sqrt{p^+}}\left[m, 0, p^+, p_x - ip_y\right] \tag{1.51}$$

$$u_-(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} 0 \\ -m \\ p_x - ip_y \\ -p^+ \end{bmatrix}, \qquad \overline{u_-}(p) = \frac{1}{\sqrt{p^+}}\left[p_x + ip_y, -p^+, 0, -m\right] \tag{1.52}$$

$$v_+(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} 0 \\ m \\ p_x - ip_y \\ -p^+ \end{bmatrix} \qquad \overline{v_+}(p) = \frac{1}{\sqrt{p^+}}\left[p_x + ip_y, -p^+, 0, m\right] \tag{1.53}$$

$$v_-(p) = \frac{1}{\sqrt{p^+}}\begin{bmatrix} p^+ \\ p_x + ip_y \\ -m \\ 0 \end{bmatrix} \qquad \overline{v_-}(p) = \frac{1}{\sqrt{p^+}}\left[-m, 0, p^+, p_x - ip_y\right] \tag{1.54}$$

Thus in the massless case we get

$$|p+>=\frac{1}{\sqrt{p^+}}\begin{bmatrix} p^+ \\ p_x+ip_y \\ 0 \\ 0 \end{bmatrix}, \qquad <p+|=\frac{1}{\sqrt{p^+}}\left[0,0,p^+,p_x-ip_y\right] \tag{1.55}$$

$$|p->=\frac{1}{\sqrt{p^+}}\begin{bmatrix} 0 \\ 0 \\ p_x-ip_y \\ -p^+ \end{bmatrix}, \qquad <p-|=\frac{1}{\sqrt{p^+}}\left[p_x+ip_y,-p^+,0,0\right] \tag{1.56}$$

$$\gamma^\mu \varepsilon^+_\mu(k,b)=\frac{\sqrt{2}\Big[|k-\rangle\langle b-|+|b+\rangle\langle k+|\Big]}{\langle bk\rangle} \tag{1.57}$$

$$\gamma^\mu \varepsilon^-_\mu(k,b)=\frac{\sqrt{2}\Big[|k+\rangle\langle b+|+|b-\rangle\langle k-|\Big]}{[kb]} \tag{1.58}$$

1.6 Fierz+Charge conjugation identity

We want to show the identity,

$$\gamma^\mu \otimes \langle A-|\gamma_\mu|B-\rangle = 2\Bigg[|A+\rangle\langle B+|+|B-\rangle\langle A-|\Bigg] \tag{1.59}$$

it is helpful to make the indices explicit, so we can rewrite this as

$$\gamma^\mu_{ij}\ \langle A-,k|(\gamma_\mu\gamma_L)_{kl}|B-,l\rangle = 2\Bigg[|A+,i\rangle\langle B+,j|+|B-,i\rangle\langle A-,j|\Bigg] \tag{1.60}$$

The indices which are normally left implicit, have been added to the bras and kets which remind us that these are four component objects. Eq. (1.60) can be written as two separate equations

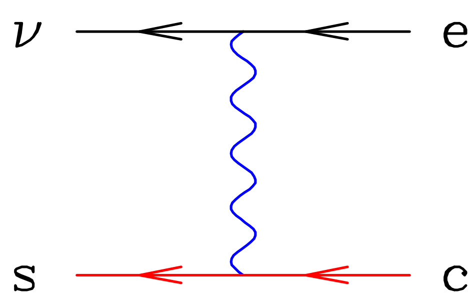
$$\begin{aligned}(\gamma^\mu\gamma_R)_{ij}\ \langle A-,k|(\gamma_\mu\gamma_L)_{kl}|B-,l\rangle &= 2\Bigg[|B-,i\rangle\langle A-,j|\Bigg]\\ (\gamma^\mu\gamma_L)_{ij}\ \langle A-,k|(\gamma_\mu\gamma_L)_{kl}|B-,l\rangle &= 2\Bigg[|A+,i\rangle\langle B+,j|\Bigg]\end{aligned} \tag{1.61}$$

It is easy to show that the first of these relations follows as a consequence of Eq. (1.6). The second would also follow as a consequence of the same equation if we are able to show that

$$\langle A-,k|(\gamma_\mu\gamma_L)_{kl}|B-,l\rangle = \langle B+,k|(\gamma_\mu\gamma_R)_{kl}|A+,l\rangle \tag{1.62}$$

We shall now show that this equation follows as a consequence of the relations obeyed under charge conjugation by massless spinors.

$$u_- = C\bar{u}_+^T, \quad u_+ = C\bar{u}_-^T \tag{1.63}$$



and hence that

$$\bar{u}_+^T = -Cu_- \tag{1.64}$$

since $C^{-1} = -C = C^T$. Thus for massless spinors we have that

$$\bar{u}_+(p_B)\gamma^\mu u_+(p_A) = -\bar{u}_-^T(p_B)C^T\gamma^\mu C\bar{u}_-^T(p_A) \tag{1.65}$$

Since from Eq.(1.38)

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T \tag{1.66}$$

this allows us to prove that

$$\langle A-|\gamma_\mu|B-\rangle = \langle B+|\gamma_\mu|A+\rangle \tag{1.67}$$

as well as the generalizations in Table 1.1.

2 Applications of spinor calculus to tree graphs

For the most part we shall exploit the remarkable simplifications which come when we consider only massless quarks.

2.1 $c\bar{s} \to W^+ \to \nu e^+$

Consider the process

$$e^- + c \leftarrow \nu + s \tag{2.1}$$

The matrix element is given by

$$\mathcal{M} = \frac{(-ig_W)^2}{2}\frac{(-i)}{P_w(s_{e\nu})}\langle \nu-|\gamma^\alpha|e-\rangle\ \langle s-|\gamma_\alpha|c-\rangle \equiv \frac{ig_W^2}{P_W(s_{e\nu})}\langle \nu s\rangle\ [ce] \tag{2.2}$$

where $P_X(p) = p^2 - m_X^2 + im_X\Gamma_X$ and that is the answer. We can identify $g_W^2/8/M_W^2 = G_F/\sqrt{2}$ by taking the low energy limit. We have assumed that all fermions, inlcuding the charmed quark are massless.

Compare the calculation performed using the traditional method with the traces. The matrix element is given by

$$\mathcal{M} \propto \bar{u}(\nu)\gamma^\alpha\gamma_L u(e)\ \bar{u}(s)\gamma_\alpha\gamma_L u(c) \tag{2.3}$$

$$
\begin{aligned}
|\mathcal{M}|^2 &= \mathrm{Tr}\{\not\nu\gamma^\alpha\gamma_L\not e\gamma^\beta\gamma_L\}\{\not s\gamma_\alpha\gamma_L\not c\gamma_\beta\gamma_L\} \\
&= 4\,\{\nu^\alpha e^\beta + \nu^\beta e^\alpha - g^{\alpha\beta} e\cdot\nu + i\epsilon^{\alpha\beta\gamma\delta}\nu_\gamma e_\delta\} \\
&\times\ \{s_\alpha c_\beta + s_\beta c_\alpha - g_{\alpha\beta} c\cdot s + i\epsilon_{\alpha\beta\rho\sigma} s^\rho c^\sigma\} \\
&= 4\, e\cdot c\; s\cdot\nu
\end{aligned}
\tag{2.4}
$$

where we have used

$$
\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\rho\sigma} = -2\Big[g^\gamma_\rho g^\delta_\sigma - g^\gamma_\sigma g^\delta_\rho\Big]
\tag{2.5}
$$

Using the spinor method the γ-matrix algebra simply disappears.

2.2 Top production and decay

If we write down amplitude we get, say for a left-handed initial quark line.

$$
\begin{aligned}
\mathcal{M} &= \frac{g_w^4}{4}\frac{g_s^2}{2}\frac{1}{D}\,(\tau^A)_{i_2 i_3}(\tau^A)_{i_1 i_4}\langle\nu - |\gamma^\mu|\bar e-\rangle\ \langle e - |\gamma^\nu|\bar\nu-\rangle\ \langle 2 - |\gamma^\alpha|3-\rangle \\
&\times\ \bar u(b)\gamma_R\gamma_\mu(\not p_1 + m_t)\gamma_\alpha(-\not p_4 + m_t)\gamma_\nu\gamma_L v(\bar b)
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
D &= P_W(\nu+\bar e)\ P_W(\bar\nu + e)\ P_t(p_1)\ P_t(p_4)\ (p_2+p_3)^2 \\
P_X(p) &= p^2 - m_X^2 + i m_X \Gamma_X
\end{aligned}
\tag{2.7}
$$

The tau's are the colour matrices normalised so that

$$
\mathrm{Tr}\tau^A\tau^B = \delta^{AB}\ .
\tag{2.8}
$$

All sorts of factors of i are missing, but since there is only one diagram, we don't care. (We are only keeping diagrams with two resonant top propagators). Using the Fierz identity twice we get

$$
\begin{aligned}
\mathcal{M} &= g_w^4\frac{g_s^2}{2}\frac{1}{D}\,(\tau^A)_{i_2 i_3}(\tau^A)_{i_1 i_4}\ \langle 2 - |\gamma^\alpha|3-\rangle \\
&\quad \bar u(b)|\nu+\rangle\ \langle\bar e + |(\not p_1 + m_t)\gamma_\alpha(-\not p_4 + m_t)|e+\rangle\ \langle\bar\nu + |v(\bar b)
\end{aligned}
\tag{2.9}
$$

If we further create massless vectors $\hat p_1, \hat p_4$ out of p_1 and p_4 by subtracting pieces proportional to the vectors $\bar e$ and e respectively

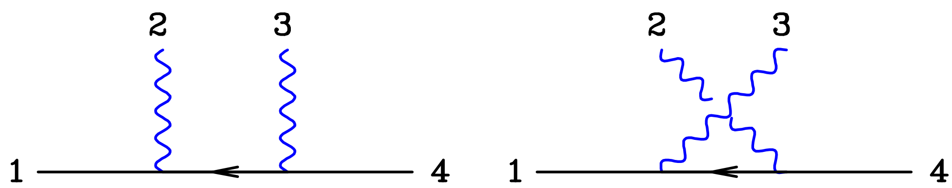
$$
p_1 = \hat p_1 + \frac{m_t^2}{2p_1\cdot\bar e}\bar e
\tag{2.10}
$$

$$
p_4 = \hat p_4 + \frac{m_t^2}{2p_4\cdot e} e
\tag{2.11}
$$

we can further simplify this to read

$$
\begin{aligned}
\mathcal{M} &= g_w^4 g_s^2\frac{1}{D}\,(\tau^A)_{i_2 i_3}(\tau^A)_{i_1 i_4} \\
&\quad \sqrt{2b\cdot\nu}\sqrt{2\bar b\cdot\bar\nu}\Big\{-[\bar e\,\hat 1]\langle\hat 1\,2\rangle[3\,\hat 4]\langle\hat 4\,e\rangle + m_t^2[\bar e\,3]\langle 2\,e\rangle\Big\}
\end{aligned}
\tag{2.12}
$$

A remarkable simple result which treats the top quarks as on shell, but preserves all the spin correlations. This calculation is given in [1]. A similar result can be produced for the $gg\to t\bar t$ process[1].



2.2.1 Homework assignment

Ref. [1] contains a serious mistake/typographical error. Find it!

2.3 $ee\gamma\gamma$

The relevant diagram is shown in Fig. 2.3. We shall consider only the case of a left-handed electron line.

$$\mathcal{M} = (-ie)^2 i \Big[\frac{\langle 1-|\not{\epsilon}_2(\not{1}+\not{2})\not{\epsilon}_3|4-\rangle}{\langle 1\,2\rangle[2\,1]} + \frac{\langle 1-|\not{\epsilon}_3(\not{1}+\not{3})\not{\epsilon}_2|4-\rangle}{\langle 1\,3\rangle[3\,1]} \Big] \tag{2.13}$$

Using Eq. (1.31) for the polarization vectors we obtain the result where both polarizations are positive

$$\mathcal{M}(1^-, 2^+_\gamma, 3^+_\gamma, 4^+_{\bar{e}}) = \frac{-2ie^2}{\langle b_2\,2\rangle\langle b_3\,3\rangle} \Big[\frac{\langle 1\,b_2\rangle[2\,1]\langle 1\,b_3\rangle[3\,4]}{\langle 1\,2\rangle[2\,1]} + \frac{\langle 1\,b_3\rangle[3\,1]\langle 1\,b_2\rangle[2\,4]}{\langle 1\,3\rangle[3\,1]} \Big] \tag{2.14}$$

Making the gauge choice $b_2 = b_3 = 1$ this gives zero. Inserting the case where the polarizations are $(+-)$ we get

$$\mathcal{M}(1^-, 2^+_\gamma, 3^-_\gamma, 4^+_{\bar{e}}) = \frac{-2ie^2}{\langle b_2\,2\rangle[3\,b_3]} \Big[\frac{\langle 1\,b_2\rangle[2\,1]\langle 1\,3\rangle[b_3\,4]}{\langle 1\,2\rangle[2\,1]} + \frac{\langle 1\,3\rangle\langle b_3+|(\not{1}+\not{3})|b_2+\rangle[2\,4]}{\langle 1\,3\rangle[3\,1]} \Big] \tag{2.15}$$

Making the gauge choice $b_2 = 1, b_3 = 4$, the first diagram gives no contribution this gives

$$\mathcal{M}(1^-_e, 2^+_\gamma, 3^-_\gamma, 4^+_{\bar{e}}) = \frac{-2ie^2}{\langle 1\,2\rangle[3\,4]} \frac{\langle 1\,3\rangle[4\,3]\langle 3\,1\rangle[2\,4]}{\langle 1\,3\rangle[3\,1]} = 2ie^2 \frac{\langle 3\,1\rangle[2\,4]}{\langle 1\,2\rangle[3\,1]} \equiv 2ie^2 \frac{[2\,4]^2}{[3\,4][3\,1]} \tag{2.16}$$

In deriving the latter formula we have used momentum conservation. Note that the result is of second degree in $|2+\rangle$ and $|3-\rangle$ as it must be for a (+-) amplitude, and also of first degree in $|1-\rangle$ and $|4+\rangle$.

In summary we find

$$\begin{aligned}
\mathcal{M}(1^-_e, 2^-_\gamma, 3^+_\gamma, 4^+_{\bar{e}}) &= 2ie^2 \frac{[2\,4]^2}{[3\,4][3\,1]} \\
\mathcal{M}(1^-_e, 2^+_\gamma, 3^-_\gamma, 4^+_{\bar{e}}) &= 2ie^2 \frac{[3\,4]^2}{[2\,4][2\,1]} \\
\mathcal{M}(1^-_e, 2^+_\gamma, 3^+_\gamma, 4^+_{\bar{e}}) &= 0 \\
\mathcal{M}(1^-_e, 2^-_\gamma, 3^-_\gamma, 4^+_{\bar{e}}) &= 0
\end{aligned} \tag{2.17}$$

Changing the helicities of all particles can be obtained by $\langle\rangle \leftrightarrow []$.

Including all four non-vanishing polarizations, we get for the spin-summed matrix element squared.

$$\sum_{hel} |\mathcal{M}|^2 = 8e^4 \Big[\frac{s_{13}}{s_{12}} + \frac{s_{12}}{s_{13}}\Big] \tag{2.18}$$

3 Relations between tree amplitudes

3.1 Colour

There are two different conventions for the colour matrices.

$$\begin{aligned}
&[t^A, t^B] = i f^{ABC} t^C, \\
&Tr(t^A, t^B) = \frac{1}{2}\delta^{AB} \\
&\sum_A t^A_{ij}\, t^A_{kl} = \frac{1}{2}\delta_{il}\, \delta_{kj} - \frac{1}{2N}\, \delta_{ij}\, \delta_{kl} \\
&f^{ABC} = -2i\mathrm{Tr}[t^A, t^B] t^C
\end{aligned} \tag{3.1}$$

It also convenient to define $\tau = t\sqrt{2}$.

$$\begin{aligned}
&[\tau^A, \tau^B] = i\sqrt{2} f^{ABC} \tau^C, \\
&Tr(\tau^A, \tau^B) = \delta^{AB} \\
&\sum_A \tau^A_{ij}\, \tau^A_{kl} = \delta_{il}\, \delta_{kj} - \frac{1}{N}\, \delta_{ij}\, \delta_{kl} \\
&f^{ABC} = -\frac{i}{\sqrt{2}}\mathrm{Tr}[\tau^A, \tau^B] \tau^C
\end{aligned} \tag{3.2}$$

In the rest of these lectures we shall adopt the second convention. Let us now consider the general colour structure for a tree-level n-gluon amplitude.

$$\mathcal{M}_n \;=\; g^{n-2} \sum_{\sigma = S_n/Z_n} Tr(\tau^{a_{\sigma(1)}} \tau^{a_{\sigma(2)}} \ldots \tau^{a_{\sigma(n)}})\; m(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n})) \tag{3.3}$$

where S_n is the set of all the permutations of n objects and Z_n is the subset of cyclic permutations. Thus for example we have,

$$\begin{aligned}
\mathcal{M}_4 \;&=\; g^2 Tr\, (\tau^{a_1}\tau^{a_2}\tau^{a_3}\tau^{a_4})\; m(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}) \\
&+\; 5 \text{ other permutations}
\end{aligned} \tag{3.4}$$

The color-stripped amplitudes are simpler than the full amplitudes because they receive contributions only from a particular ordering of the gluons. Colour stripped amplitudes have poles only in invariants made out of adjacent momenta.

We can do a similar decomposition for the amplitude with two external quarks and $n-2$ gluons.

$$\mathcal{M}_n = g^{n-2} \sum_{\sigma=S_{(n-2)}} (\tau^{a_{\sigma(2)}} \dots \tau^{a_{\sigma(n-1)}})_{i_n}^{i_1}\, m(1_q^{\lambda_1}, \sigma(2^{\lambda_2}) \dots, \sigma((n-1)^{\lambda_{n-1}}), n_{\bar{q}}^{\lambda_n}) \tag{3.5}$$

3.2 The quark-gluon scattering process

The Feynman rules for QCD are shown in Fig. 3.1.

As a further example of the use of spinor techniques we will consider the process shown in Fig. (3.2).

$$0 \to q(p_1) + g(p_2) + g(p_3) + \bar{q}(p_4) \tag{3.6}$$

where the momentum labels are indicated in brackets

- We shall first decompose the amplitude interms of colour ordered sub-amplitudes. The sub-amplitudes are gauge invariant

$$M(p_1, h_1; p_2, h_2; p_3, h_3; p_4, h_4) = \tag{3.7}$$
$$g^2\ \tau^{A_2}\tau^{A_3} m_1(h_1, h_2, h_3, h_4) + g^2\ \tau^{A_3}\tau^{A_2} m_2(h_1, h_3, h_2, h_4) \tag{3.8}$$

$$\sum |M|^2 = g^4 \frac{N^2-1}{N} \Big[N^2(|m_1|^2 + |m_2|^2) - |m_1 + m_2|^2 \Big] \tag{3.9}$$

- subleading terms in $N \equiv$ QED. Colour ordered amplitudes are obtained by using the relation

$$f^{ABC} = -\frac{i}{\sqrt{2}} \mathrm{Tr}[\tau^A, \tau^B]\tau^C \tag{3.10}$$

  to simplify diagrams containing the three and four gluon vertices and by also using the $SU(N)$ crossing relation

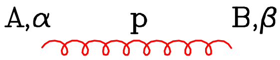
$$\sum_A \tau_{ij}^A\ \tau_{kl}^A = \delta_{il}\ \delta_{kj} - \frac{1}{N}\ \delta_{ij}\ \delta_{kl}\ . \tag{3.11}$$

- Simplify the calculation by astute choice of the reference momenta. calculate m_1 only, the positive helicity quark line

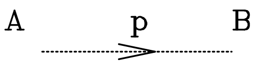
$$m_2^{(a)} = \frac{-i}{2} \langle p_1 + | \not{\varepsilon}_2 \frac{(\not{p}_1 + \not{p}_2)}{\langle p_2 p_1 \rangle [p_1 p_2]} \not{\varepsilon}_3 | p_4 + \rangle \tag{3.12}$$

$$m_q^{(b)} = 0 \tag{3.13}$$

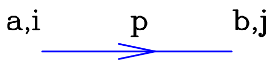
$$\begin{aligned} m_q^{(c)} &= \frac{-i}{\langle p_2 p_3 \rangle [p_3 p_2]} \\ &\Big[\varepsilon_2 \cdot \varepsilon_3\ \langle p_1, + | \not{p}_3 | p_4, + \rangle + \varepsilon_3 \cdot p_2\ \langle p_1, + | \not{\varepsilon}_2 | p_4, + \rangle \\ &- \varepsilon_2 \cdot p_3\ \langle p_1, + | \not{\varepsilon}_3 | p_4, + \rangle \Big] \end{aligned} \tag{3.14}$$



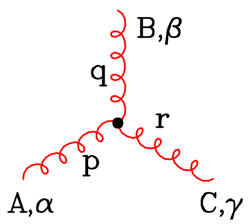
$$\delta^{AB}\ \left[-g^{\alpha\beta}+(1-\lambda)\,\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\ \frac{i}{(p^{2}+i\epsilon)}$$

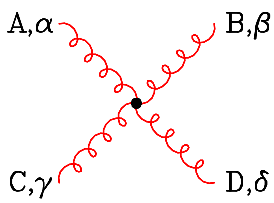


$$\delta^{ab}\ \frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

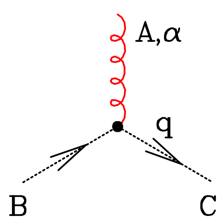


$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

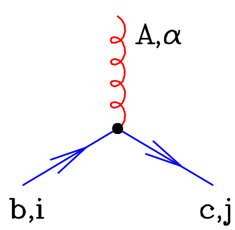
(all momenta incoming)



$$\begin{aligned}
&-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]\\
&-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]\\
&-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]
\end{aligned}$$



$$g\ f^{ABC}q^{\alpha}$$



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

Figure 3.1: Feynman rules for QCD

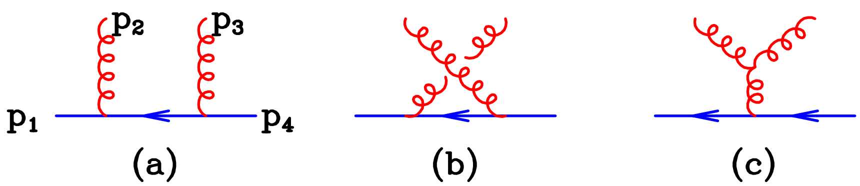


Figure 3.2: Feynman diagrams for the process $q \to ggq$.

When the helicities of the two gluons are the same we shall choose the two reference momenta b_2, b_3 to be the same; it then follows that $\varepsilon_2 \cdot \varepsilon_3 = 0$. For the positive helicity case we choose $b_2 = b_3 = p_4$ so that

$$\not{\varepsilon}_2{}^+|p_4,+\rangle = \not{\varepsilon}_3{}^+|p_4,+\rangle = 0 \tag{3.15}$$

None of the diagrams contribute to $m_1(+,+,+,-)$. A similar simplification occurs for the $m_1(+,-,-,-)$. choosing $b_2 = b_3 = p_1$.

- For the mixed helicity case it is convenient to choose $b_2 = p_3$ and $b_3 = p_2$, so that $\varepsilon_2 \cdot \varepsilon_3 = \varepsilon_2 \cdot p_3 = \varepsilon_3 \cdot p_2 = 0$. The full result comes from the first diagram alone

$$m_1(1_q^+, 2_g^+, 3_g^+, 4_{\bar{q}}^-) = 0 \tag{3.16}$$

$$m_1(1_q^+, 2_g^-, 3_g^-, 4_{\bar{q}}^-) = 0 \tag{3.17}$$

$$m_1(1_q^+, 2_g^+, 3_g^-, 4_{\bar{q}}^-) = -i\frac{\langle 34\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle} \tag{3.18}$$

$$m_1(1_q^+, 2_g^-, 3_g^+, 4_{\bar{q}}^-) = i\frac{[13]^3[34]}{[12][23][34][14]} \tag{3.19}$$

- We have calculated the quark gluon scattering matrix element in an non-Abelian theory, with no net contribution from the diagram involving the three gluon vertex. Its effect is completely fixed by gauge invariance.
- The non-zero amplitudes m_2 can be obtained by Bose symmetry.

$$m_2(1_q^+, 2_g^+, 3_g^-, 4_{\bar{q}}^-) = i\frac{[12]^3[24]}{[13][32][24][14]} \tag{3.20}$$

$$m_2(1_q^+, 2_g^-, 3_g^+, 4_{\bar{q}}^-) = -i\frac{\langle 24\rangle^3\langle 12\rangle}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 14\rangle} \tag{3.21}$$

$$\tag{3.22}$$

Because of the parity invariance of the strong interactions we have that

$$m(h_1, h_2, h_3, h_4) = m^*(-h_1, -h_2, -h_3, -h_4) \tag{3.23}$$

The final result for sum of the squared amplitudes

$$\sum_{c,h} |M|^2 = 2g^4 \frac{(N^2-1)}{N}\Big[N^2(1-2\frac{tu}{s^2}) - 1\Big](\frac{u}{t} + \frac{t}{u}) \tag{3.24}$$

where we have defined $s = (p_2+p_3)^2, t = 2p_1.p_2, u = 2p_1.p_3$.

3.3 Three point vertices

By direct insertion of the Feynman rules we have

$$A(1_g^+, 2_{\bar{q}}^-, 3_q^+) \;=\; -i\sqrt{2}g(t^{a_1})_{i_3 i_2}\frac{[3\,1]\langle b\,2\rangle}{\langle b\,1\rangle} = -i\sqrt{2}g(t^{a_1})_{i_3 i_2}\frac{[1\,3]^2}{[2\,3]} \tag{3.25}$$

$$A(1_g^-, 2_{\bar{q}}^-, 3_q^+) \;=\; -i\sqrt{2}g(t^{a_1})_{i_3 i_2}\frac{[3\,b]\langle 1\,2\rangle}{[1\,b]} = -i\sqrt{2}g(t^{a_1})_{i_3 i_2}\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle} \tag{3.26}$$

where $[t^{a_1}, t^{a_2}] = if^{a_1 a_2 a_3} t^{a_3}$. By the same token we have that, (choosing the gauge vector b the same for all polarizations,

$$\begin{aligned} A(1_g^+, 2_g^-, 3_g^+) &= gf^{a_1 a_2 a_3}\Big[\epsilon_2\cdot\epsilon_3 2\epsilon_1\cdot p_2 - \epsilon_1\cdot\epsilon_2 2\epsilon_3\cdot p_2\Big] \\ &= \sqrt{2}gf^{a_1 a_2 a_3}\frac{\langle 2\,b\rangle^2[3\,1][b\,2]}{[2\,b]\langle b\,3\rangle\langle b\,1\rangle} = \sqrt{2}gf^{a_1 a_2 a_3}\frac{[1\,3]^3}{[2\,3][1\,2]} \\ &= -i\sqrt{2}g(F^{a_1})_{a_3 a_2}\frac{[1\,3]^3}{[2\,3][1\,2]} \end{aligned} \tag{3.27}$$

where $(F^{a_1})_{a_3 a_2} = if^{a_1, a_2, a_3}$, so that $[F^{a_1}, F^{a_2}] = if^{a_1 a_2 a_3}F^{a_3}$.

$$\begin{aligned} A(1_g^-, 2_g^-, 3_g^+) &= gf^{a_1 a_2 a_3}\Big[\epsilon_3\cdot\epsilon_1 2\epsilon_2\cdot p_3 - \epsilon_2\cdot\epsilon_3 2\epsilon_1\cdot p_3\Big] \\ &= \sqrt{2}gf^{a_1 a_2 a_3}\frac{[3\,b]^2}{[b\,1][b\,2]\langle b\,3\rangle}\Big[\langle 1\,b\rangle\langle 2\,3\rangle - \langle 2\,b\rangle\langle 1\,3\rangle\Big] \\ &= \sqrt{2}gf^{a_1 a_2 a_3}\frac{[3\,b]^2\langle 1\,2\rangle}{[b\,1][b\,2]} = -\sqrt{2}gf^{a_1 a_2 a_3}\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 1\,3\rangle} \\ &+ \; i\sqrt{2}g(F^{a_1})_{a_3 a_2}\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 1\,3\rangle} \end{aligned} \tag{3.28}$$

These three point vertices can only be defined for complex momenta, since we have

$$p_1\cdot p_2 = p_2\cdot p_3 = p_3\cdot p_1 = 0 \tag{3.29}$$

Thus for example we have that in Eq. (3.25), $\langle 1\,2\rangle = \langle 2\,3\rangle = \langle 3\,1\rangle = 0$.

If we were to take the fermions in the adjoint representation

$$(t^{a_1})_{i_3 i_2} \to (F^{a_1})_{a_3 a_2} \tag{3.30}$$

we would have that

$$\begin{aligned} A(1_g^-, 2_g^-, 3_g^+) &= -A(1_g^-, 2_{\bar{q}}^-, 3_q^+)\frac{\langle 1\,2\rangle}{\langle 1\,3\rangle} \\ A(1_g^+, 2_g^-, 3_g^+) &= +A(1_g^+, 2_{\bar{q}}^-, 3_q^+)\frac{[1\,3]}{[1\,2]} \end{aligned} \tag{3.31}$$

3.4 Susy relations between amplitudes

We would like to use supersymmetry to relate tree amplitudes in QCD graphs. The material in this section is taken almost verbatim from Mangano and Parke, ref.[2]. We encourage the students to look at ref. [2] for a definitive treatment.

Since QCD is not a supersymmetric theory, why should QCD amplitudes be related by supersymmetric transformations? The answer is that once the colour degrees of freedom have been stripped off, there is no difference between a massless quark and massless gluino.

We will obtain relations based on the commutator of a global supersymmetric charge with a string of field operators. We assume that the supersymmetric charge annihilates the vacuum, so that

$$\begin{aligned} 0 &= \langle [Q, \phi_1\phi_2\phi_3\dots\phi_n]\rangle \\ &= \langle [Q,\phi_1]\,\phi_2\phi_3\dots\phi_n\rangle + \langle \phi_1\,[Q,\phi_2]\,,\phi_3\dots\phi_n\rangle + \dots \end{aligned} \tag{3.32}$$

So we need the commutators of the SUSY charges with gluons g and gluinos Λ. We multiply the supersymmetric charge Q by Grassman spinor parameter to obtain $Q(\eta)$. Then $Q(\eta)$ acts on the doublet (g,Λ) as follows[3, 2]

$$[Q(\eta), g^{\pm}(p)] = \mp\,\Gamma^{\pm}(p,\eta)\;\Lambda^{\pm}, \tag{3.33}$$

$$[Q(\eta), \Lambda^{\pm}(p)] = \mp\,\Gamma^{\mp}(p,\eta)\;g^{\pm}. \tag{3.34}$$

$\Gamma^{\pm}(p,\eta)$ is a complex function linear in the anticommuting c-number components of η and satisfies:

$$\Gamma^{+}(p,\eta) = [\Gamma^{-}(p,\eta)]^{*} = \bar\eta\;u_{-}(p), \tag{3.35}$$

The function Γ has its form constrained Jacobi identity

$$[[Q(\eta),Q(\xi)],\phi] + [[Q(\xi),\phi],Q(\eta)] + [[\phi,Q(\eta)],Q(\xi)] = 0 \tag{3.36}$$

Since $[Q(\eta),Q(\xi)] = 2i\bar\eta \not{P}\xi$ we derive the relation

$$\Gamma^{+}(k,\eta)\Gamma^{-}(k,\xi) + \Gamma^{-}(k,\eta)\Gamma^{+}(k,\xi) = -2i\bar\eta\not{P}\xi \tag{3.37}$$

with $u_{-}(p)$ a negative helicity spinor satisfying the massless Dirac equation with momentum p. Because of the arbitrariness in choosing the supersymmetry parameter η, we choose this to be a negative helicity spinor obeying the Dirac equation with an arbitrary massless momentum k times a Grassmann variable θ. θ implies that $\Gamma^{\pm}(p,\eta)$ anti-commutes with the fermion creation and annihilation operators and commutes with the bosonic operators.

$$\Gamma^{+}(p,k) \equiv \Gamma^{+}(p,\eta(k)) = \theta\langle k+|p-\rangle \equiv \theta\;[kp]. \tag{3.38}$$

As a notation, we choose to label the supersymmetry charge $Q(\eta)$ with the momentum k characterising the parameter η: $Q(k) = Q[\eta(k)]$. We can now operate with $Q(\eta)$ on s string of gluon and gluino fields and take the vacuum expectation value.

$$\begin{aligned} 0 = \langle [Q, \Lambda_1^{+} g_2^{+} g_3^{+}\dots g_n^{+}]\rangle &= -\Gamma^{-}(p_1,k)\;A(g_1^{+},g_2^{+},\dots,g_n^{+}) \\ &+\Gamma^{+}(p_2,k)\;A(\Lambda_1^{+},\Lambda_2^{+},\dots,g_n^{+}) + \dots + \Gamma^{+}(p_n,k)\;A(\Lambda_1^{+},g_2^{+},\dots,\Lambda_n^{+}). \end{aligned} \tag{3.39}$$

Since all of the amplitudes involving gluinos Λ vanish, we conclude that gluon amplitudes with all helicities the same must be zero.

To prove a similar theorem for amplitudes with one helicity flip, let us consider the following identity:

$$\begin{aligned} 0 &= \langle [Q, \Lambda_1^+ g_2^- g_3^+ \dots g_n^+] \rangle = \\ & - \Gamma^-(p_1, k)\ A(g_1^+, g_2^-, \dots, g_n^+) + \Gamma^-(p_2, k)\ A(\Lambda_1^+, \Lambda_2^-, \dots, g_n^+). \end{aligned} \tag{3.40}$$

In the above equation we have dropped all of the amplitudes with both fermions having the same helicity, which must vanish by helicity conservation. Eq. (3.40) must be satisfied for any choice of the vector k, and in particular we can then choose $k = p_2$, proving that the gluonic amplitude must vanish, or $k = p_1$, thus proving the vanishing of the amplitudes with the fermion pair.

For the maximally helicity violating amplitude we have $(g_1^-, g_2^-, g_3^+, \dots, g_n^+)$, with two negative-helicity gluons and $n-2$ positive-helicity gluons where all of the particles are outgoing.

$$\begin{aligned} & \Gamma^-(p_1, k) A(\Lambda_1^-, g_2^-, \Lambda_3^+, g_4^+, \dots, g_n^+) + \Gamma^-(p_2, k) A(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \dots, g_n^+) \\ - & \ \Gamma^-(p_3, k) A(g_1^-, g_2^-, g_3^+, \dots, g_n^+) = 0. \end{aligned} \tag{3.41}$$

Choosing, for example, $k = p_1$ we therefore obtain the following relation:

$$A(g_1^-, g_2^-, g_3^+, \dots, g_n^+) = \frac{\langle 1\,2 \rangle}{\langle 1\,3 \rangle} A(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \dots, g_n^+) \tag{3.42}$$

We can now apply this theorem to the specific four parton case that we have calculated in the section. For the two quark, two gluon case we obtain from Eq. (3.18) (performing the swap $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$ that

$$m_1(1_g^-, 2_{\bar{q}}^-, 3_q^+, 4_g^+) = -i \frac{\langle 12 \rangle^3 \langle 31 \rangle}{\langle 34 \rangle \langle 41 \rangle \langle 12 \rangle \langle 32 \rangle} \tag{3.43}$$

Hence the result for the four gluon helicity amplitude is,

$$m_1(1_g^-, 2_g^-, 3_g^+, 4_g^+) = -i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{3.44}$$

This is the only non-vanishing amplitude for the four gluon case.

3.5 BCFW

We have seen that Supersymmetry has allowed us to calculate simple gluonic amplitudes by relating them to amplitudes involving gluinos (ie quarks) and gluons, with the same number of external legs. We now want to illustrate the technique for something more ambitious, namely sewing together on-shell amplitudes to produce tree graphs with larger numbers of legs[4]. Rather than describing the on-shell recursion in detail, we shall decribe

	ggg	$g\bar{q}q$
$m(1^-,2^-,3^+)$	$\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 1\,3\rangle}$	$-\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle}$
$m(1^+,2^-,3^+)$	$\frac{[1\,3]^3}{[1\,2][2\,3]}$	$\frac{[1\,3]^2}{[2\,3]}$

Table 3.1: Relations between color-stripped amplitudes for different three parton processes

	$gggg$	$g\bar{q}qg$	$\bar{Q}\bar{q}qQ$
$m(1^-,2^-,3^+,4^+)$	$\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}$	$-\frac{\langle 1\,2\rangle^2\langle 1\,3\rangle}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}$	$-\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\langle 4\,1\rangle}$
$m(1^-,2^-,3^-,4^+)$	0	0	0

Table 3.2: Relations between color-stripped amplitudes for different four parton processes

the opposite process, effectively decomposing the 4-gluon MHV amplitude (Eq. (3.44)) into the product of two on-shell three gluon momenta, but at shifted momenta.

We define the shifted momenta $1 \to \hat{1}, 4 \to \hat{4}$ as

$$\begin{aligned} \tilde{\lambda}_1 &\to \tilde{\lambda}_1 + z\tilde{\lambda}_4 \\ \lambda_4 &\to \lambda_4 - z\lambda_1 \end{aligned} \tag{3.45}$$

Because $\tilde{\lambda}_j \equiv |j], \lambda_j \equiv |j\rangle$, in the bra ket notation this is equivalent to

$$\begin{aligned} k_1^\mu &\to k_{\hat{1}}^\mu(z) = k_1^\mu + \tfrac{z}{2}\langle 1|\gamma^\mu|4] \\ k_4^\mu &\to k_{\hat{4}}^\mu(z) = k_4^\mu - \tfrac{z}{2}\langle 1|\gamma^\mu|4] \end{aligned} \tag{3.46}$$

This makes the external momenta k_1 and k_4 complex, but preserves overall momentum conservation. In addition the vectors $k_{\hat{1}}, k_{\hat{4}}$ are still lightlike. We indicate these shifted z-dependent momenta by putting a hat over them.

Furthermore we have the following behaviour of the spinor products under this shift

$$\begin{aligned} 2k_{\hat{1}} \cdot k_2 &= \langle 2|\not{k}_{\hat{1}}|2] = \langle 1\,2\rangle\ \{[2\,1] + z[2\,4]\} \\ 2k_{\hat{4}} \cdot k_3 &= \langle 3|\not{k}_{\hat{4}}|3] = \{\langle 4\,3\rangle - z\langle 1\,3\rangle\}[3\,4]\} \\ 2k_{\hat{1}} \cdot k_{\hat{4}} &= 2k_1 \cdot k_4 = \langle 1\,4\rangle[4\,1] \end{aligned} \tag{3.47}$$

Now consider our 4 gluon result, Eq. (3.44)

$$A(1^-,2^-,3^+,4^+) = -i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \tag{3.48}$$

but with the shifted momenta,

$$\mathcal{A}(z) \equiv A(\hat{1}^-, 2^-, 3^+, \hat{4}^+) = -i \frac{\langle \hat{1}\, 2\rangle^4}{\langle \hat{1}\, 2\rangle \langle 2\, 3\rangle \langle 3\, \hat{4}\rangle \langle \hat{4}\, \hat{1}\rangle} \tag{3.49}$$

But from Eq. (3.47) we have that

$$\langle \hat{1}\, 2\rangle = \langle 1\, 2\rangle, \;\; \langle \hat{1}\, \hat{4}\rangle = \langle 1\, 4\rangle, \;\; \langle 3\, \hat{4}\rangle = \langle 3\, 4\rangle - z\langle 3\, 1\rangle \tag{3.50}$$

and any shift in the $[i\, j]$ are irrelevant since our formula only depends on $\langle i\, j\rangle$. Thus we find

$$\mathcal{A}(z) = -i \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle (\langle 3\, 4\rangle - z\langle 3\, 1\rangle) \langle 4\, 1\rangle} \tag{3.51}$$

Now if we divide this function $\mathcal{A}(z)$ by z we get two poles, at $z = 0$ and at $z = \langle 3\, 4\rangle / \langle 3\, 1\rangle$. The residue at $z = 0$ corresponds to the original unshifted amplitude. The other residue corresponds to the case where an intermediate propagator is on shell, because we have that $(k_3 + k_4(z))^2 = \langle 3\, \hat{4}\rangle [\hat{4}\, 3] = (\langle 3\, 4\rangle - z\langle 3\, 1\rangle)[4\, 3] = 0$. Thus if we take contour integral

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} \mathcal{A}(z) \tag{3.52}$$

If the circle at infinity gives no contribution, we obtain

$$\mathcal{A}(z=0) = -\sum_{poles} \mathrm{Res}\left[\frac{\mathcal{A}(z)}{z}\right] \tag{3.53}$$

In this case we only have one pole (apart from the pole at $z = 0$), and the contributions of the two residues are equal and opposite because the contour at infinity gives zero. Since the second pole corresponds to the vanishing of an intermediate propagator, the full result factorizes about this pole into the product of two three point amplitudes multiplied by a scalar propagator.

We have thus managed to write the 4 point amplitude as a product of the two on-shell three point momenta at complex shifted values

$$\begin{aligned} \mathcal{A}(z=0) &= -A_3(-\hat{k}_{34}^-, 3^+, \hat{4}^+) \left[\mathrm{Res} \frac{-i}{z\langle 3\, \hat{4}\rangle [\hat{4}\, 3]}\right] A_3(\hat{1}^-, 2^-, \hat{k}_{34}^+) \\ \mathcal{A}(z=0) &= A_3(-\hat{k}_{34}^-, 3^+, \hat{4}^+) \frac{-i}{\langle 3\, 4\rangle [4\, 3]} A_3(\hat{1}^-, 2^-, \hat{k}_{34}^+) \end{aligned} \tag{3.54}$$

Putting in explicit expressions for the three point amplitudes we obtain,

$$\begin{aligned} \mathcal{A}(z=0) &= \frac{[3\, 4]^4}{[3\, \hat{4}][4\, \hat{k}_{34}][\hat{k}_{34}\, 3]} \frac{i}{\langle 3\, 4\rangle [4\, 3]} \frac{-\langle 1\, 2\rangle^4}{\langle \hat{1}\, 2\rangle \langle 2\, \hat{k}_{34}\rangle \langle \hat{k}_{34}\, \hat{1}\rangle} \\ &= \frac{-[3\, 4]^3 \langle 1\, 2\rangle^3}{\langle 1|(\hat{k}_{34})|3] \langle 2|(\hat{k}_{34})|4]} \frac{-i}{\langle 3\, 4\rangle [4\, 3]} \\ &= -i \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle 3\, 4\rangle \langle 4\, 1\rangle} \end{aligned} \tag{3.55}$$

In performing this last step we have used the identities that

$$\begin{aligned} \langle 1|\hat{k}_{34}|3] &= \langle 1|(3+4)|3] = \langle 1\,4\rangle[4\,3] \\ \langle 2|(\hat{k}_{34})|4] &= \langle 2|(3+4)|4] = \langle 2\,3\rangle[3\,4] \end{aligned} \tag{3.56}$$

which are evident from the shifts Eq. (3.46) which do not change λ_1 or $\tilde{\lambda}_4$. We have thus recovered our original MHV amplitude, but we have shown that is can be calculated as a product of three point amplitudes with shifted external momenta, connected by a scalar propagator. This illustrates the essence of the BCFW on-shell recursion relations.

The result for the n-point MHV amplitude can be derived using BCFW recursion.

$$M(1^-,2^-,3^+\ldots,(n-1)^+,n^+) = -i\frac{\langle 1\,2\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n-1\,n\rangle)\langle n\,1\rangle} \tag{3.57}$$

4 One loop diagrams: the traditional approach

4.1 Scalar Integrals

$$\begin{aligned} & I_1^d(m_1^2) = \frac{\mu^{4-d}}{i\pi^{\frac{d}{2}}r_\Gamma}\int d^d l\,\frac{1}{(l^2-m_1^2+i\varepsilon)}\,, \\ & I_2^d(p_1^2;m_1^2,m_2^2) = \frac{\mu^{4-d}}{i\pi^{\frac{d}{2}}r_\Gamma}\int d^d l\,\frac{1}{(l^2-m_1^2+i\varepsilon)((l+q_1)^2-m_2^2+i\varepsilon)}\,, \\ & I_3^d(p_1^2,p_2^2,p_3^2;m_1^2,m_2^2,m_3^2) = \frac{\mu^{4-d}}{i\pi^{\frac{d}{2}}r_\Gamma} \\ & \times\int d^d l\,\frac{1}{(l^2-m_1^2+i\varepsilon)((l+q_1)^2-m_2^2+i\varepsilon)((l+q_2)^2-m_3^2+i\varepsilon)}\,, \\ & I_4^d(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mu^{4-d}}{i\pi^{\frac{d}{2}}r_\Gamma} \\ & \times\int d^d l\,\frac{1}{(l^2-m_1^2+i\varepsilon)((l+q_1)^2-m_2^2+i\varepsilon)((l+q_2)^2-m_3^2+i\varepsilon)((l+q_3)^2-m_4^2+i\varepsilon)}\,, \end{aligned} \tag{4.1}$$

where $q_n \equiv \sum_{i=1}^n p_i$ and $q_0 = 0$ and $s_{ij} = (p_i+p_j)^2$. For the purposes of this paper we take the masses in the propagators to be real. Near four dimensions we use $d = 4-2\epsilon$. (For clarity the small imaginary part which fixes the analytic continuations is specified by $+i\,\varepsilon$). μ is a scale introduced so that the integrals preserve their natural dimensions, despite excursions away from $d = 4$. We have removed the overall constant which occurs in d-dimensional integrals

$$r_\Gamma \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\Gamma(1-\epsilon)} + \mathcal{O}(\epsilon^3) = 1-\epsilon\gamma+\epsilon^2\left[\frac{\gamma^2}{2}-\frac{\pi^2}{12}\right]+\mathcal{O}(\epsilon^3)\,. \tag{4.2}$$

Feynman parameter identities are also useful; we have

$$
\begin{aligned}
\frac{1}{A^\alpha\, B^\beta \ldots F^\phi} &= \frac{\Gamma(\alpha+\beta+\ldots\phi)}{\Gamma(\alpha)\Gamma(\beta)\ldots\Gamma(\phi)} \\
&\times \int_0^1 da_1\, da_2 \ldots da_n\, \delta(1-a_1-a_2\ldots-a_n) \\
&\times \frac{a_1^{\alpha-1}\, a_2^{\beta-1}\ldots a_n^{\phi-1}}{(Aa_1+Ba_2+\ldots+Fa_n)^{\alpha+\beta+\ldots+\phi}}
\end{aligned}
\tag{4.3}
$$

We shall process this integral using the fundamental formula for one-loop integrals given here,

$$
\begin{aligned}
&\frac{1}{i\pi^{\frac{d}{2}}}\int d^dk \frac{(-k^2)^r}{\Big[-k^2+C-i\varepsilon\Big]^m} = \\
&[C-i\varepsilon]^{2+r-m-\epsilon}\,\frac{\Gamma(r+d/2)}{\Gamma(d/2)}\frac{\Gamma(m-r-2+\epsilon)}{\Gamma(m)}.
\end{aligned}
\tag{4.4}
$$

After Feynman parametrization and integration over d^Dl, we have for the triangle and box integrals

$$
I_3^D(p_1^2,p_2^2,p_3^2;m_1^2,m_2^2,m_3^2) = -\frac{\mu^{2\epsilon}\Gamma(1+\epsilon)}{r_\Gamma}\prod_{i=1}^{3}\int_0^1 da_k\,\frac{\delta(1-\sum_k a_k)}{\Big[\sum_{i,j}a_ia_jY_{ij}-i\varepsilon\Big]^{1+\epsilon}}\,, \tag{4.5}
$$

$$
I_4^D(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mu^{2\epsilon}\Gamma(2+\epsilon)}{r_\Gamma}\prod_{i=1}^{4}\int_0^1 da_k\,\frac{\delta(1-\sum_k a_k)}{\Big[\sum_{i,j}a_ia_jY_{ij}-i\varepsilon\Big]^{2+\epsilon}}\,, \tag{4.6}
$$

where Y is the so-called modified Cayley matrix

$$
Y_{ij} \equiv \frac{1}{2}\Big[m_i^2+m_j^2-(q_{i-1}-q_{j-1})^2\Big]\,. \tag{4.7}
$$

4.1.1 Dimensional Regularisation

In the intermediate stages of the calculation we must introduce some regularisation procedure to control these divergences. The most effective regulator is the method of dimensional regularisation which continues the dimension of space-time to $d = 4-2\epsilon$ dimensions [5]. This method of regularisation has the advantage that the Ward Identities of the theory are preserved at all stages of the calculation. Integrals over loop momenta are performed in d dimensions with the help of the following formula,

$$
\begin{aligned}
&\int\frac{d^dk}{(2\pi)^d}\frac{(-k^2)^r}{\Big[-k^2+C-i\varepsilon\Big]^m} = \\
&\frac{i(4\pi)^\epsilon}{16\pi^2}\,[C-i\varepsilon]^{2+r-m-\epsilon}\,\frac{\Gamma(r+d/2)}{\Gamma(d/2)}\frac{\Gamma(m-r-2+\epsilon)}{\Gamma(m)}.
\end{aligned}
\tag{4.8}
$$

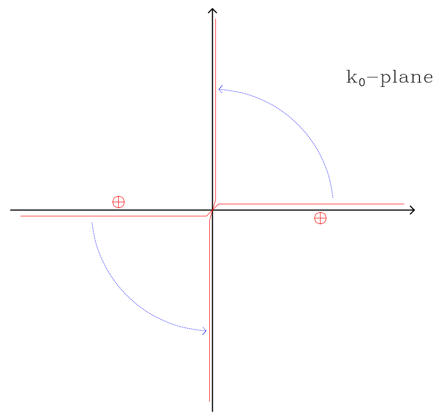


Figure 4.1: Wick rotation in the complex k_0 plane

To demonstrate Eq. (4.8), we first perform a Wick rotation of the k_0 contour anti-clockwise. This is dictated by the $i\varepsilon$ prescription, since for real C the poles coming from the denominator of Eq. (4.8) lie in the second and fourth quadrant of the k_0 complex plane as shown in Fig. 4.1. Thus by anti-clockwise rotation we encounter no poles. After rotation by an angle $\pi/2$, the k_0 integral runs along the imaginary axis in the k_0 plane, $(-i\infty < k_0 < i\infty)$. In order to deal only with real quantities we make the substitution $k_0 = i\kappa_d, k_j = \kappa_j$ for all $j \neq 0$ and introduce $|\kappa| = \sqrt{\kappa_1^2 + \kappa_2^2 \ldots + \kappa_d^2}$. We obtain a d-dimensional Euclidean integral which may be written as,

$$\begin{aligned}\int d^d\kappa\ f(\kappa^2) \quad &= \quad \int d|\kappa|\ f(\kappa^2)\ |\kappa|^{d-1} \sin^{d-2}\theta_{d-1} \sin^{d-3}\theta_{d-2} \ldots \\ &\quad \times \sin\theta_2\ d\theta_{d-1} d\theta_{d-2} \ldots d\theta_2 d\theta_1. \end{aligned} \tag{4.9}$$

This formula is best proved by induction. The range of the angular integrals is $0 \leq \theta_i \leq \pi$ except for $0 \leq \theta_1 \leq 2\pi$. The angular integrations, which only give an overall factor, can be performed using

$$\int_0^\pi d\theta\ \sin^d\theta = \sqrt{\pi}\frac{\Gamma\Big(\frac{(d+1)}{2}\Big)}{\Gamma\Big(\frac{(d+2)}{2}\Big)}. \tag{4.10}$$

We therefore find that the left hand side of Eq. (4.8) can be written as,

$$\frac{2i}{(4\pi)^{d/2}\Gamma\Big(d/2\Big)} \int_0^\infty d|\kappa|\ \frac{|\kappa|^{d+2r-1}}{\Big[\kappa^2 + C\Big]^m}. \tag{4.11}$$

This last integral can be reduced to a Beta function, (see Table 4.1)

$$\int_0^\infty dx \frac{x^s}{\Big[x^2 + C\Big]^m} = \frac{\Gamma\big(\frac{(s+1)}{2}\big)}{2} \frac{\Gamma\big(m - s/2 - 1/2\big)}{\Gamma\big(m\big)} C^{s/2+1/2-m} \tag{4.12}$$

which demonstrates Eq. (4.8).

$\Gamma(z) = \int_0^\infty \, dt \, e^{-t} t^{z-1}$
$z\Gamma(z) = \Gamma(z+1)$
$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2})$
$\Gamma(n+1) = n!$ for n a positive integer
$\Gamma(1) = 1, \;\; \Gamma(\frac{1}{2}) = \sqrt{\pi}$
$\Gamma'(1) = -\gamma_E, \;\; \gamma_E \approx 0.577215$
$\Gamma''(1) = \gamma_E^2 + \frac{\pi^2}{6}$
$B(a,b) = \int_0^1 dx \, x^{a-1}(1-x)^{b-1}$
$B(a,b) = \int_0^\infty dt \, \frac{t^{a-1}}{(1+t)^{a+b}}$ for Re $a, b > 0$
$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Table 4.1: Useful properties of the Γ and related functions

4.1.2 Landau conditions

The necessary conditions for eqs. (4.5,4.6) to contain a singularity are due to Landau [6, 7]. If we introduce the bilinear form d derived from the modified Cayley matrix, cf. Eq.(4.6)

$$d = \sum_{i,j} a_i a_j Y_{ij}\,, \tag{4.13}$$

eqs. (4.5, 4.6) contain singularities if $d = 0$ and one of the following conditions is satisfied for all values of j

$$\textbf{either } a_j = 0 \textbf{ or } \frac{\partial d}{\partial a_j} = 0\,. \tag{4.14}$$

Since d is a homegeneous function of the a_i of degree two, we have that

$$a_i \frac{\partial d}{\partial a_j} = 2d \tag{4.15}$$

So the conditions in Eq. (4.14) also imply that $d = 0$.

4.1.3 Soft and collinear divergences

The class of solution, which is of interest here, is the case where the external virtualities and internal masses have fixed values and the Landau conditions have solutions for arbitrary values of the other external invariants, s_{ij}. Only these solutions will lead to soft and collinear divergences which are relevant for next-to-leading order calculations.

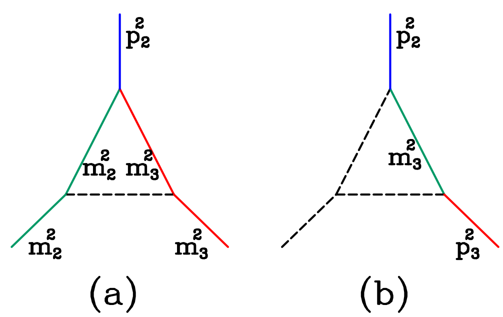


Figure 4.2: Examples of triangle diagrams with divergences.

As an example we consider the triangle shown in Fig. (4.2a) which contains a soft singularity. In this case the denominator is given by

$$D = \sum_{i,j} a_i a_j Y_{ij} = (m_2^2 + m_3^2 - p_2^2)a_2 a_3 + m_2^2 a_2^2 + m_3^2 a_3^2\,. \tag{4.16}$$

This expression satisfies the Landau conditions for $a_2 = a_3 = 0$ and a_1 arbitrary. A second example is the triangle shown in Fig. (4.2b) which contains a collinear singularity. In this case the denominator reads

$$D = \sum_{i,j} a_i a_j Y_{ij} = (m_3^2 - p_2^2)a_2 a_3 + (m_3^2 - p_3^2)a_1 a_3 + m_3^2 a_3^2\,, \tag{4.17}$$

which satisfies the Landau conditions for $a_3 = 0$ and a_1, a_2 arbitrary.

From the Landau conditions it follows that a necessary condition for a soft or collinear singularity is that for at least one value of the index i [8]

$$Y_{i+1\,i+1} = Y_{i+1\,i+2} = Y_{i+1\,i} = 0\,, \quad \text{soft singularity}\,, \tag{4.18}$$

$$Y_{i\,i} = Y_{i+1\,i+1} = Y_{i\,i+1} = 0\,, \quad \text{collinear singularity}\,. \tag{4.19}$$

The indices in eqs. (4.18, 4.19) should be interpreted mod N, where N is the number of external legs. Thus the structure of the Cayley matrices for integrals having a soft or collinear divergence is as follows

$$Y_{\text{soft}} = \begin{pmatrix} \dots & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots \\ \dots & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad Y_{\text{collinear}} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{4.20}$$

In order to have a divergence, we must have at least one internal mass equal to zero, i.e. at least one vanishing diagonal element of Y.

4.1.4 Scalar Integrals

Here we give an example of the result a scalar integral regularized by dimensional regularization, $d = 4 - 2\epsilon$.

$$\begin{aligned} I_4^D(0,0,0,0;s_{12},s_{23};0,0,0,0) &= \frac{\mu^{2\epsilon}}{s_{12}s_{23}} \\ \times \quad \Bigg\{ \frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon} + (-s_{23}-)^{-\epsilon}\Big) &- \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) - \pi^2 \Bigg\} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.21}$$

This result is taken from [9]. A basis set of scalar one-loop integrals has been presented in ref. [10]. In addition there is a numerical code, called QCDLoop that returns the numerical value of any one-loop integral as a Laurent series in $1/\epsilon$. Thus the problem of one-loop integrals can be cansidered as completely solved, at least as far as NLO calculations are concerned.

4.2 Passarino-Veltman

Tensor loop integrals can be reduced to sums of scalar integrals using the Passarino-Veltman decomposition. As an example consider the form factor decomposition of a simple rank 1 triangle diagram.

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{4.22}$$

$$\int \frac{d^nl}{(2\pi)^n}\frac{l^\mu l^\nu}{(l^2-m_1^2)((l+p)^2-m_2^2)((l+q)^2-m_3^2)}$$

$$= \begin{pmatrix} p^\mu p^\nu & q^\mu q^\nu & (p^\mu q^\nu + q^\mu p^\nu) & g^{\mu\nu} \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{22} \\ C_{12} \\ C_{00} \end{pmatrix} \tag{4.23}$$

We can solve for C_1, C_2 by contracting with the external momenta, p, q.

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l\cdot p] \\ [2l\cdot q] \end{pmatrix} = G\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p\cdot p & 2p\cdot q \\ 2p\cdot q & 2q\cdot q \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{4.24}$$

where the notation is $[2l\cdot p] = \int \frac{d^nl}{(2\pi)^n}\frac{2l\cdot p}{l^2(l+p)^2(l+q)^2}$ by expressing $2l\cdot p, (2l\cdot q)$ as a sum of denominators $2l\cdot p = (l+p)^2 - l^2 - p^2$ we can express R_1, R_2 as a sum of scalar integrals Solving we get

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = G^{-1}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{4.25}$$

G is the Gram matrix

$$G = \begin{pmatrix} 2p\cdot p & 2p\cdot q \\ 2p\cdot q & 2q\cdot q \end{pmatrix}, \quad \Delta_2(p,q) = |G| = 4(p^2q^2 - (p\cdot q)^2) \tag{4.26}$$

$$G^{-1} = \frac{\begin{pmatrix} 2q\cdot q & -2p\cdot q \\ -2p\cdot q & 2p\cdot p \end{pmatrix}}{\Delta_2(p,q)} \tag{4.27}$$

Thus the solution is $C = G^{-1}R$ This solution appears to have a problem when $p \parallel q$ and the Gram determinant vanishes; the original tensor integral had no special problems when $p \parallel q$.

G can be diagonalized by an orthogonal transformation $G = ODO^T$, $D = \mathrm{diag}\{\lambda_+, \lambda_-\}$ Defining modified form factors C' and inhomogeneous terms, R' by the transformations $C' = O^TC,\ \ R' = O^TR$, we have the solution:-

$$\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 1/\lambda_+ & 0 \\ 0 & 1/\lambda_- \end{pmatrix}\begin{pmatrix} R_1' \\ R_2' \end{pmatrix} \tag{4.28}$$

In the singular region one of the eigenvalues, say λ_- will vanish

4.2.1 Singular region

Now consider the approach to the singular region by setting $q_\mu = \kappa p_\mu + \delta_\mu$ and keeping only the leading terms in δ. The eigenvalues are

$$\lambda_+ = 2p^2(1+\kappa^2), \ \ \lambda_- = \frac{2(\delta^2p^2 - (\delta\cdot p)^2)}{p^2(1+\kappa^2)}, \ \ |G| = 4(\delta^2p^2 - (\delta\cdot p)^2) \tag{4.29}$$

λ_- vanishes like $O(\delta^2)$ The matrix of eigenvectors is

$$O \sim \frac{1}{\sqrt{1+\kappa^2}}\begin{pmatrix} 1 - \frac{\kappa\delta.p}{p^2(1+\kappa^2)} & \kappa + \frac{\kappa\delta.p}{p^2(1+\kappa^2)} \\ \kappa + \frac{\kappa\delta.p}{p^2(1+\kappa^2)} & -1 + \frac{\kappa\delta.p}{p^2(1+\kappa^2)} \end{pmatrix} \tag{4.30}$$

$$\int \frac{d^n l}{(2\pi)^n}\frac{l^\mu}{l^2(l+p)^2(l+q)^2} = \left(\begin{array}{cc} p'_\mu & q'_\mu \end{array} \right) \left(\begin{array}{c} C'_1 \\ C'_2 \end{array} \right)$$
$$= \left(\begin{array}{cc} p'_\mu & q'_\mu \end{array} \right) \left(\begin{array}{cc} 1/\lambda_+ & 0 \\ 0 & 1/\lambda_- \end{array} \right) \left(\begin{array}{c} R'_1 \\ R'_2 \end{array} \right)$$

The momentum corresponding to the singular eigenvalue is

$$q'_\mu = -\delta_\mu + \frac{\delta \cdot p\kappa(1+\kappa)}{p^2(1+\kappa^2)} = O(\delta) \tag{4.31}$$

$$R'_2 \sim \kappa[2l\cdot p] - [2l\cdot q] \sim O(\delta) \tag{4.32}$$

As expected the result for the tensor integral is finite in the limit $\delta \to 0$, but the vanishing of R'_2 is not manifest; it is realized as a property of a combination of scalar integrals. One approach would be to work in the primed basis, which would thus differ for every phase space point. (Numerical problems halved?)

4.3 Rational terms by PV reduction

The rational part is related to the ultraviolet behavior of the theory; the naive expectation is that the better the UV behavior, the "smaller" the rational part. When the integral is free from the rational part, it is said to be "cut-constructible". A natural expectation is that the rational part is absent in UV-finite integrals. As we explain below, this expectation turns out to be wrong; the correct result is that a Feynman N-point integral is cut constructible, provided that tensor rank, r, of the integral satisfies the following condition [11]

$$r < \max\{(N-1), 2\}\,. \tag{4.33}$$

The condition is illustrated in Fig. 4.3. If this condition is violated the integral will contain rational parts. Explicitly, Eq. (4.33) implies that the UV finite rank-two four-point function is cut-constructible, whereas the UV-finite rank-three four-point function is not.

In this section we give an proof of the condition that an integral has to satisfy for being cut-constructible, Eq. (4.33). This proof is based on the Passarino-Veltman reduction. We will proceed case-by-case for the two-, three- and four-point integrals which occur in a renormalizable theory. The extension to higher point integrals will be performed at the end. We first note that the Passarino-Veltman decomposition described in Section **??** and **??**, yields the coefficients of the scalar integrals D_0, C_0, B_0, A_0 for arbitrary values of the number of dimensions. Since the rational terms are related to UV singularities they will show up at the end of the reduction as terms of the form

$$\text{Rational terms } \sim \epsilon B_0(p, m_1, m_2)\,, \tag{4.34}$$

because B_0 is the only UV divergent scalar integral. Such terms can only arise if the reduction involves the dimensional parameter D. This means that integrals of rank r less than two will always be cut-constructible, since their reduction coefficients are always D independent. Ultraviolet divergent integrals of rank two or greater (e.g. $D_{iiii}, C_{iii}, C_{ii}, B_{ii}$)

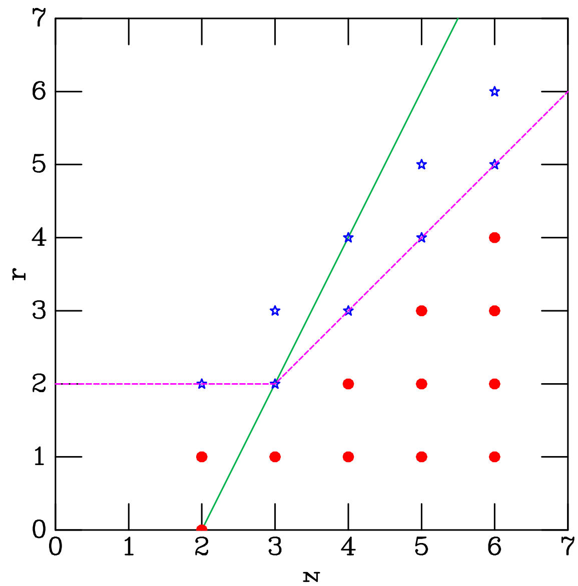


Figure 4.3: Diagram showing tensor N-point integrals of rank r. Integrals shown by bullets (red) are cut-constructible, integrals denoted by stars (blue) contain rational terms. The UV finite integrals lie beneath the solid (green) line, whereas the cut constructible integrals lie beneath the dashed (purple) line.

will on the contrary give rise to rational parts. Thus it only remains to discuss the ultraviolet finite integrals D_{iii}, D_{ii}. The integral D_{iii} contains a UV-divergent integrals of rank greater than two in its reduction paths, $D_{iii} \to C_{ii}$, see Table **??** and hence will have a rational part. This leaves the special case D_{ii}, a finite integral which can contain a UV divergent integral in its reduction path, namely B_0. However since the starting integral is UV finite, the UV poles all cancel. Moreover the coefficients of B_0 are all ϵ-independent, since the only D dependence enters through D_{00}, which does not contain B_0 in its reduction path. Hence the rank-two, four point integral is cut constructible.

In a renormalizable theory the higher point functions are not UV divergent. Moreover the most UV singular terms in their reduction paths reduce both N and r by one unit. Therefore the reduction paths of these UV finite integrals can only generate a rational part if the rank of the integral has $r \geq N-1$. This observation extends Eq. (4.33) to N greater than 4.

4.4 The importance of the van Neerven - Vermaseren basis

The material in this section is taken from a forthcoming review, [12]. On-shell scattering amplitudes in gauge field theories are gauge-invariant. A practical version of this statement is that an on-shell scattering amplitude evaluated with the polarization vector of a particular gauge boson substituted by the momentum of that gauge boson must vanish, when all the other gauge bosons have physical polarizations. This provides both a constraint on the form of the amplitude and a powerful check of the computation. However, it is well-known that in complicated cases that involve high-point scattering amplitudes, demonstrating this cancellation analytically is non-trivial. One reason why such complications arise is the dimensionality of space-time since it implies that for high-point amplitudes external momenta are not linearly independent. In four dimensions, the "dimensionality constraint" can be stated in the form of the Schouten identity

$$q_\alpha \epsilon_{\beta\gamma\delta\lambda} = q_\beta \epsilon_{\alpha\gamma\delta\lambda} + q_\gamma \epsilon_{\beta\alpha\delta\lambda} + q_\delta \epsilon_{\beta\gamma\alpha\lambda} + q_\lambda \epsilon_{\beta\gamma\delta\alpha}, \tag{4.35}$$

which follows from the vanishing of the totally antisymmetric rank-five tensor in four dimensions.

Since these constraints are not implemented in the Passarino-Veltman procedure, it is usually not easy to demonstrate gauge cancellations in that framework. Vermaseren and Oldenborgh pointed out that constraints related to the dimensionality of space-time can be conveniently implemented if the loop momentum is written as a linear combination of appropriate basis vectors [13]. We call this set of vectors the van Neerven - Vermaseren basis [14]. In addition to making the gauge-independence of one-loop amplitude more transparent, this basis turns out to be very convenient for an easy proof of Eq.(**??**) which states that any one-loop integral can be written as a linear combination of four-, three-, two- and one-point scalar functions. Moreover, the van Neerven - Vermaseren basis proved to be very fruitful for understanding a number of important results that concern the reduction of tensor integrals and the applicability of the generalized unitarity. Below we summarize some of the results obtained using the van Neerven - Vermaseren basis.

First, in four dimensions, simple algorithms were derived for the reduction of tensor integrals to the linear combination of box, triangle, bubble and tadpole scalar integrals. The number of terms generated in this process has been reduced in comparison with the standard Passarino-Veltman reduction procedure. Second, using the van Neerven - Vermaseren basis, it is straightforward to show that in four dimensions the scalar five-point Feynman integral is given by a linear combination of scalar box integrals [15, 14]. Third, using the van Neerven - Vermaseren basis it is easy to understand that in four dimensions the *integrand* of any one-loop Feynman diagram in any renormalizable theory is given by a linear combination of quadruple, triple-, double- and single-pole rational functions with the numerator of a very restrictive form. Finally, employing the van Neerven - Vermaseren decomposition, it is straightforward to find the loop momenta that satisfy quadruple, triple-, double-, and single-cut on-shell conditions. These features of the van Neerven - Vermaseren basis make it important for the construction of the technique of generalized D-dimensional unitarity. Because of that the following subsections are devoted to its detailed explanation.

4.5 Physical space of inflow momenta and transverse space

We consider a N-particle scattering amplitude in a renormalizable quantum field theory in D-dimensional space-time. Such an amplitude can be computed from relevant Feynman

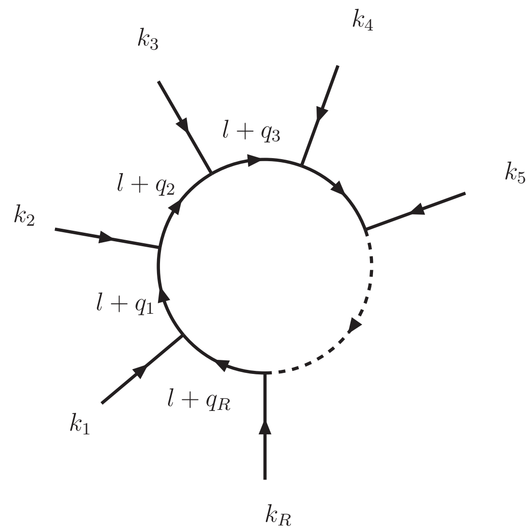


Figure 4.4: Generic diagram

diagrams, each given by an integral over the loop momentum l of an integrand function. We study one of these Feynman diagrams and imagine that it has R loop-momentum-dependent propagators. The integrand $\mathcal{I}_N$ is a rational function of the loop momentum l given by the product of R l-dependent scalar inverse propagators d_i and a polynomial in l of rank $r_l \leq R$.

$$\mathcal{I}_N(p_1, p_2, \ldots, p_N|l) = \frac{\mathcal{N}_{\mathcal{I}}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_R} \,. \tag{4.36}$$

The amplitude has a set of R outflow momenta, $k_1, \ldots, k_R$. The outflow momenta are either equal to the external momenta p_i, or are given by their linear combinations

$$d_i = (l + q_i)^2 - m_i^2 \,, \quad k_i = q_i - q_{i-1}, \;\; k_i = \sum_{j=1}^{N} \alpha_{ij} p_j \,, \quad \sum_{i=1}^{R} k_i = 0, \tag{4.37}$$

where α_{ij} are diagram-specific numbers. We call the vector space spanned by the outflow momenta the *physical space*. We emphasize that the dimensionality D_P of the physical space changes from diagram to diagram. Accounting for the momentum conservation $\sum\limits_{i=1}^{R} k_i = 0$, we obtain

$$D_P = \min(D, R - 1), \tag{4.38}$$

which implies that for $R < D$, the dimensionality of the physical space is smaller that the dimensionality of space-time. Authors of Ref. [14] exploit this observation by defining an orthonormal basis – the van Neerven - Vermaseren basis – that naturally describes the D-dimensional space split into a D_P-dimensional physical space and a D_T-dimensional transverse space. Dimensionalities of various spaces satisfy obvious constraints

$$D = D_P + D_T, \;\; D_P = \min\,(D, R - 1), \;\; D_T = \max\,(0, D - R + 1). \tag{4.39}$$

If $R > D$, the transverse space is zero-dimensional.

To define the van Neerven - Vermaseren basis we introduce the generalized Kronecker symbol [13] [1]

$$\delta^{\mu_1\mu_2\cdots\mu_R}_{\nu_1\nu_2\cdots\nu_R} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \dots & \delta^{\mu_1}_{\nu_R} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \dots & \delta^{\mu_2}_{\nu_R} \\ \vdots & \vdots & & \vdots \\ \delta^{\mu_R}_{\nu_1} & \delta^{\mu_R}_{\nu_2} & \dots & \delta^{\mu_R}_{\nu_R} \end{vmatrix} , \tag{4.40}$$

the compact notation

$$\delta^{k\mu_2\cdots\mu_R}_{\nu_1 q\cdots\nu_R} \equiv \delta^{\mu_1\mu_2\cdots\mu_R}_{\nu_1\nu_2\cdots\nu_R} k_{\mu_1} q^{\nu_2} , \tag{4.41}$$

and the R-particle Gram determinant

$$\Delta(k_1, k_2, \cdots, k_R) = \delta^{k_1 k_2\cdots k_R}_{k_1 k_2\cdots k_R} . \tag{4.42}$$

Note that for $R \geq D+1$ the generalized Kronecker delta vanishes. For the special case $D = R$ the Kronecker delta factorizes into the product of two Levi-Civita tensors $\delta^{\mu_1\mu_2\cdots\mu_R}_{\nu_1\nu_2\cdots\nu_R} = \varepsilon^{\mu_1\mu_2\cdots\mu_R}\varepsilon_{\nu_1\nu_2\cdots\nu_R}$. If the number of outflow momenta is small, we can write the Kronecker deltas explicitly

$$\begin{aligned} \delta^{k_1k_2}_{q_1\mu} &= k_1\cdot q_1\,\delta^{k_2}_{\mu} - k_{1\mu}\delta^{k_2}_{q_1}, = k_1\cdot q_1\,k_{2\mu} - k_2\cdot q_1\,k_{1\mu}, \\ \delta^{k_1k_2k_3}_{q_1q_2q_3} &= k_1\cdot q_1\,\delta^{k_2k_3}_{q_2q_3} - k_1\cdot q_2\,\delta^{k_2k_3}_{q_1q_3} + k_1\cdot q_3\,\delta^{k_2k_3}_{q_1q_2} \\ &= \ k_1\cdot q_1\,(k_2\cdot q_2\,k_3\cdot q_3 - k_2\cdot q_3\,k_3\cdot q_2) \\ &- k_1\cdot q_2\,(k_2\cdot q_1\,k_3\cdot q_3 - k_2\cdot q_3\,k_3\cdot q_1) \\ &+ k_1\cdot q_3\,(k_2\cdot q_1\,k_3\cdot q_2 - k_2\cdot q_2\,k_3\cdot q_1). \end{aligned} \tag{4.43}$$

We can use the Kronecker δ-symbol to construct the van Neerven - Vermaseren basis for the physical space D_p. We define the basis vectors

$$v_i^{\mu}(k_1,\dots,k_{D_P}) \equiv \frac{\delta^{k_1\dots k_{i-1}\mu k_{i+1}\dots k_{D_P}}_{k_1\dots k_{i-1}k_i k_{i+1}\dots k_{D_P}}}{\Delta(k_1,\dots,k_{D_P})} , \tag{4.44}$$

with the properties $v_i\cdot k_j = \delta_{ij}$ for $j \leq D_P$. When $R \leq D$ we also need to define the projection operator onto the transverse space

$$w_{\mu}{}^{\nu}(k_1,\dots,k_{R-1}) \equiv \frac{\delta^{k_1\cdots k_{R-1}\nu}_{k_1\dots k_{R-1}\mu}}{\Delta(k_1,\dots,k_{R-1})} , \tag{4.45}$$

[1] This notation is closely related to the asymmetric Gram determinant notation of ref. [16],

$$G\begin{pmatrix} k_1 & \cdots & k_R \\ & & \\ q_1 & \cdots & q_R \end{pmatrix} = \delta^{k_1k_2\cdots k_R}_{q_1q_2\cdots q_R} .$$

with the properties $w_\mu{}^\mu = D_T = D+1-R$, $k_i^\mu\, w_{\mu\nu} = 0$ and $w^\mu{}_\alpha w^{\alpha\nu} = w^{\mu\nu}$. We note that $w^{\mu\nu}$ is the metric tensor of the transverse subspace, amenable to decomposition

$$w^{\mu\nu} = \sum_{i=1}^{D+1-R} n_i^\mu n_i^\nu . \tag{4.46}$$

The $D+1-R$ orthonormal basis vectors of the transverse space n_i have the properties $n_i \cdot n_j = \delta_{ij},\ n_i \cdot k_j = n_i \cdot v_j = 0$. The full metric tensor decomposition in the van Neerven-Vermaseren basis is given by

$$g^{\mu\nu} = \sum_{i=1}^{D_P} k_i^\mu v_i^\nu + w^{\mu\nu} = \sum_{i=1}^{D_P} k_i^\mu v_i^\nu + \sum_{i=1}^{D_T} n_i^\mu n_i^\nu \ . \tag{4.47}$$

Note that the right hand side of this equation is, actually, a symmetric tensor since, by explicitly writing the generalized Kronecker delta-function using k_i vectors, one can show that the following equation holds

$$\sum_{i=1}^{D_P} k_i^\mu v_i^\nu = \sum_{i=1}^{D_P} k_i^\nu v_i^\mu . \tag{4.48}$$

For the case $D = R$, the only basis vector of the one-dimensional transverse space is proportional to the Levi-Civita tensor. For the cases $R < D$ we can explicitly construct the basis vectors that fulfill all the requirements. As an example, if $D = 4$ and $R = 4$, we get

$$\begin{aligned}
& v_1^\mu(k_1,k_2,k_3) = \frac{\delta^{\mu k_2 k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)},\ v_2^\mu(k_1,k_2,k_3) = \frac{\delta^{k_1 \mu k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)}, \\
& v_3^\mu(k_1,k_2,k_3) = \frac{\delta^{k_1 k_2 \mu}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)}; \\
& w_\mu{}^\nu(k_1,k_2,k_3) = \frac{\delta^{k_1 k_2 k_3 \nu}_{k_1 k_2 k_3 \mu}}{\Delta(k_1,k_2,k_3)} = n_{1\mu} n_1{}^\nu = \frac{\varepsilon_{k_1 k_2 k_3 \mu}\varepsilon^{k_1 k_2 k_3 \nu}}{\Delta(k_1,k_2,k_3)} \ .
\end{aligned} \tag{4.49}$$

In applications of the van Neerven-Vermaseren basis, it is often needed to write the loop momentum l as a linear combination of the basis vectors for a particular graph with the denominator factors $d_1, d_2, ...d_R$. The denominators are given by $d_i = (l+q_i)^2 - m_i^2$ and the outflow momenta read $k_i = q_i - q_{i-1}$. By contracting in the loop momentum with the metric tensor given in Eq. (4.47) we obtain the loop momentum decomposition in the van Neerven - Vermaseren basis

$$l^\mu = \sum_{i=1}^{D_P} (l \cdot k_i)\, v_i^\nu + \sum_{i=1}^{D_T} (l \cdot n_i)\ n_i^\nu \ . \tag{4.50}$$

Using the identity

$$l \cdot k_i = \frac{1}{2}\left[d_i - d_{i-1} - \left(q_i^2 - m_i^2\right) + \left(q_{i-1}^2 - m_{i-1}^2\right)\right] \ , \tag{4.51}$$

we find

$$l^\mu = V_R^\mu + \frac{1}{2}\sum_{i=1}^{D_P}(d_i - d_{i-1})\, v_i^\mu + \sum_{i=1}^{D_T}(l \cdot n_i)\, n_i^\mu \ , \tag{4.52}$$

where $d_0 = d_R$, $m_0 = m_R$ and

$$V_R^\mu = -\frac{1}{2}\sum_{i=1}^{D_P}\Big((q_i^2 - m_i^2) - (q_{i-1}^2 - m_{i-1}^2)\Big)\, v_i^\mu \ . \tag{4.53}$$

As an illustration of this procedure, we explicitely give the loop-momentum decomposition in two cases. The first example concerns the five-point function in four dimensions, so that $D = 4$ and $R = 5$. We derive

$$\begin{aligned} l^\mu &= V_5^\mu + \frac{1}{2}(d_1 - d_5)\, v_1^\mu + \frac{1}{2}(d_2 - d_1)\, v_2^\mu \\ &\quad + \frac{1}{2}(d_3 - d_2)\, v_3^\mu + \frac{1}{2}(d_4 - d_3)\, v_4^\mu, \\ V_5^\mu &= -\frac{1}{2}(q_1^2 - q_5^2 - m_1^2 + m_5^2)\, v_1^\mu - \frac{1}{2}(q_2^2 - q_1^2 - m_2^2 + m_1^2)\, v_2^\mu \\ &\quad - \frac{1}{2}(q_3^2 - q_2^2 - m_3^2 + m_2^2)\, v_3^\mu - \frac{1}{2}(q_4^2 - q_3^2 - m_4^2 + m_3^2)\, v_4^\mu \ . \end{aligned} \tag{4.54}$$

Similarly, for a three-point function in four dimensions $D = 4$ and $R = 3$. We obtain

$$\begin{aligned} l^\mu &= V_3^\mu + \frac{1}{2}(d_1 - d_3)\, v_1^\mu + \frac{1}{2}(d_2 - d_1)\, v_2^\mu + (l \cdot n_1)\, n_1^\mu + (l \cdot n_2)\, n_2^\mu, \\ V_3^\mu &= -\frac{1}{2}(q_1^2 - q_3^2 - m_1^2 + m_3^2)\, v_1^\mu - \frac{1}{2}(q_2^2 - q_1^2 - m_2^2 + m_1^2)\, v_2^\mu \ . \end{aligned} \tag{4.55}$$

We note that if the number of ourflow momenta R exceeds the dimensionality of space-time D, the decomposition of the loop momentum into the van Neerven - Vermaseren basis may be used to prove that the $D + m$ point functions $m \geq 1$ can all be written as linear combinations of the D-point functions. We will show an example of this in the next Section. Finally, we emphasize that the van Neerven - Vermaseren basis allows us to include the unitarity constraints without resorting to spinor-helicity formalism, which is often used in analytic calculations. By avoiding the spinor-helicity formalism, the method can be used in computations with massive internal particles, where the mass can be either real or complex-valued.

5 OPP and Numerical Unitarity

Analytic calculations in four dimensions require considerable algebraic effort. I have therefore chosen to present the basic ideas using two examples in two dimensions where the algebraic burden is lighter.

5.1 Reduction of a two-dimensional triangle to a sum of bubbles

In this section I will show that a scalar triangle in two dimensions, can be reduced to a sum of bubbles[12]. The demonstration is essentially identical to the demonstration that a four-dimensional pentagon can be reduced to a sum of five boxes. So results such a these lie at the basis of the general OPP expansion Eq. (**??**) considered in the next section. We define

$$\begin{aligned} v_1^\mu &= \frac{\delta^{\mu q_2}_{q_1 q_2}}{\Delta_2}, \\ v_2^\mu &= \frac{\delta^{q_1 \mu}_{q_1 q_2}}{\Delta_2}, \end{aligned} \tag{5.1}$$

where $\Delta_2 = \delta^{q_1 q_2}_{q_1 q_2}$ is the two-dimensional Gram determinant, so that $v_i . q_j = \delta_{ij}$. We want to examine a scalar triangle in 2 dimensions

$$I_3 = \int d^2 l \frac{1}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2)((l+q_2)^2 - m_2^2)} = \int d^2 l \frac{1}{d_0 d_1 d_2} \tag{5.2}$$

Now expansion for l is

$$l^\mu = l \cdot q_1 \ v_1^\mu + l \cdot q_2 \ v_2^\mu \,. \tag{5.3}$$

Hence we have that,

$$l^2 = l \cdot q_1 \ v_1 \cdot l + l \cdot q_2 \ v_2 \cdot l \,, \tag{5.4}$$

so the two dimensional metric tensor can be written as,

$$g^{\mu\nu}_{(2)} = v_1^\mu q_1^\nu + v_2^\mu q_2^\nu \,. \tag{5.5}$$

Defining

$$r_1 = q_1^2 + m_0^2 - m_1^2, \quad r_2 = q_2^2 + m_0^2 - m_2^2 \,, \tag{5.6}$$

we can write Eq. (5.4) in terms of the denominators of Eq. (5.2)

$$2d_0 + 2m_0^2 - l \cdot v_1 (d_1 - d_0 - r_1) - l \cdot v_2 (d_2 - d_0 - r_2) = 0 \,. \tag{5.7}$$

Dividing by $d_0 d_1 d_2$ we see that terms with d_2 and d_1 drop out upon integration. So we get,

$$\frac{2d_0 + 2m_0^2 + d_0(l \cdot v_1 + l \cdot v_2) + r_1 l \cdot v_1 + r_2 l \cdot v_2}{d_0 d_1 d_2} = 0 \,. \tag{5.8}$$

Now consider the term where d_0 cancels. We may define a shifted momentum $l' = l + q_1$. Hence

$$l \cdot v_1 + l \cdot v_2 = l' \cdot v_1 - q_1 \cdot v_1 + l' \cdot v_2 - q_1 \cdot v_2 = l' \cdot v_1 - 1 + l' \cdot v_2 \tag{5.9}$$

The term linear in l' vanishes after integration, $(q_2 - q_1) \cdot (v_1 + v_2)$, so we get,

$$\frac{d_0 + 2m_0^2 + r_1 l \cdot v_1 + r_2 l \cdot v_2}{d_0 d_1 d_2} = \frac{d_0 + 2m_0^2 + l \cdot w}{d_0 d_1 d_2} = 0 \,, \tag{5.10}$$

where $w = r_1 v_1 + r_2 v_2$. We now replace the metric tensor in Eq.(5.10) using Eq. (5.5) and get that

$$\frac{d_0 + 2m_0^2 + l \cdot q_1 \; v_1 \cdot w + l \cdot q_2 \; v_2 \cdot w}{d_0 d_1 d_2} = 0 \,. \tag{5.11}$$

Finally

$$\frac{d_0 + 2m_0^2 + \frac{1}{2}(d_1 - d_0 - r_1) \; v_1 \cdot w + \frac{1}{2}(d_2 - d_0 - r_2) \; v_2 \cdot w}{d_0 d_1 d_2} = 0 \,, \tag{5.12}$$

which proves the relation. Collecting terms we get

$$\frac{d_0(2 - (v_1 \cdot w + v_2 \cdot w)) + (4m_0^2 - w^2) + d_1 \; v_1 \cdot w + d_2 \; v_2 \cdot w}{d_0 d_1 d_2} = 0 \,. \tag{5.13}$$

This demonstrates that the scalar triangle can we written as a sum of bubbles. The explicit solution is,

$$\int d^2 l \frac{1}{d_0 d_1 d_2} = \frac{1}{4m_0^2 - w^2} \left[(v_1 \cdot w + v_2 \cdot w - 2) \; I_{12} - v_1 \cdot w \; I_{02} - v_2 \cdot w \; I_{01} \right] \tag{5.14}$$

where

$$I_{ij} = \int d^2 l \; \frac{1}{d_i d_j} \tag{5.15}$$

5.1.1 Reduction of triangle at the integrand level

Let us further process Eq. (5.7) but without performing the integral. We have that

$$2d_0 + 2m_0^2 - l \cdot v_1 (d_1 - d_0) - l \cdot v_2 (d_2 - d_0) + l \cdot w = 0 \,, \tag{5.16}$$

but we can write

$$\begin{aligned} l \cdot w &= l \cdot q_1 \; v_1 \cdot w + l \cdot q_2 \; v_2 \cdot w \\ &= \frac{1}{2}[d_1 - d_0 - r_1] \; v_1 \cdot w + \frac{1}{2}[d_2 - d_0 - r_2] \; v_2 \cdot w \\ &= \frac{1}{2}[d_1 - d_0] \; v_1 \cdot w + \frac{1}{2}[d_2 - d_0] \; v_2 \cdot w - \frac{1}{2} w^2 \,. \end{aligned} \tag{5.17}$$

Collecting terms again we get (at integrand level!),

$$\begin{aligned} \frac{1}{d_0 d_1 d_2} &= \frac{1}{4m_0^2 - w^2} \Bigg[(v_1 \cdot w + v_2 \cdot w - 4 - 2l \cdot v_1 - 2l \cdot v_2) \; \frac{1}{d_1 d_2} \\ &- \; (v_1 \cdot w - 2l \cdot v_1) \; \frac{1}{d_0 d_2} - (v_2 \cdot w - 2l \cdot v_2) \; \frac{1}{d_0 d_1} \Bigg] \,. \end{aligned} \tag{5.18}$$

This expression essentially proves the form of the integrand needed for the OPP reduction. In two dimensions we only have to include bubble integrals and the most general form of the integrand is

$$\frac{\mathcal{N}}{(l^2 - m_0^2)((l + q_1)^2 - m_1^2)} = \frac{b_0 + b_1 (n_{q_1} \cdot l)}{d_0 d_1} \tag{5.19}$$

where we have given the example of the bubble with the first two denominators, d_0, d_1. Since we are dealing with a scalar triangle $\mathcal{N}$ is really just the denominator, d_2.

$$\mathcal{N} = \frac{1}{d_2} \; . \tag{5.20}$$

We shall chose to expand the momentum in terms of v_1 and v_2. The two constraints $d_0 = d_1 = 0$ fix the momentum l_c to be

$$l_c^\mu = -\frac{1}{2} r_1 v_1^\mu + \beta v_2^\mu \tag{5.21}$$

where β satisfies the equation coming from $d_0(l_c) = 0$

$$\beta^2 v_2^2 - r_1 \beta v_1 \cdot v_2 + v_1^2 r_1^2/4 - m_0^2 = 0; \tag{5.22}$$

leading to

$$\beta_\pm = \frac{r_1 v_1 \cdot v_2 \pm \sqrt{r_1^2 (v_1 \cdot v_2)^2 - r_1^2 v_1^2 v_2^2 + 4 m_0^2 v_2^2}}{2 v_2^2} \tag{5.23}$$

so that

$$\begin{aligned} \beta_+ + \beta_- &= \frac{r_1 v_1 \cdot v_2}{v_2^2} \\ 4\beta_+ \beta_- &= \frac{1}{v_2^2}(r_1^2 v_1^2 - 4 m_0^2) \end{aligned} \tag{5.24}$$

Therefore we obtain,

$$d_2(l_c) = 2 l_c \cdot q_2 + r_2 = r_2 + 2\beta_\pm \tag{5.25}$$

Therefore we have that

$$\begin{aligned} \frac{1}{2}\left[\frac{1}{d_2(l_c^+)} + \frac{1}{d_2(l_c^-)}\right] &= \frac{r_2 + \beta_+ + \beta_-}{r_2^2 + 2 r_2(\beta_+ + \beta_-) + 4\beta_+\beta_-} = \frac{v_2 \cdot w}{w^2 - 4 m_0^2} \\ \frac{1}{2}\left[\frac{1}{d_2(l_c^+)} - \frac{1}{d_2(l_c^-)}\right] &= \frac{\beta_- - \beta_+}{r_2^2 + 2 r_2(\beta_+ + \beta_-) + 4\beta_+\beta_-} = -\frac{2 l_c . v_2}{w^2 - 4 m_0^2} \end{aligned} \tag{5.26}$$

Thus our final results for b_0 and b_1 are,

$$\begin{aligned} b_0 &= \frac{1}{2}\left[\frac{1}{(l_c^+ + q_2)^2 - m_2^2)} + \frac{1}{(l_c^- + q_2)^2 - m_2^2)}\right] \\ b_1 &= \frac{1}{2}\left[\frac{1}{(l_c^+ + q_2)^2 - m_2^2)} - \frac{1}{(l_c^- + q_2)^2 - m_2^2)}\right] . \end{aligned} \tag{5.27}$$

are thus in agreement with Eq. (5.18).

5.2 Reduction of a rank-two two-point function

Our next two-dimensional example concerns the reduction of a rank-two two-point function using van Neerven - Vermaseren basis. Consider an integrand given by

$$\mathcal{I}(k,m_1,m_2) = \frac{(\hat{n}\cdot l)^2}{d_1 d_2}, \tag{5.28}$$

where $d_1 = l^2 - m_1^2$, $d_2 = (l+k)^2 - m_2^2$, $\hat{n}\cdot k = 0, k^2 \neq 0$ and $\hat{n}^2 = 1$. Because of the projection onto $\hat{n}$, the momentum l in the numerator in Eq. (5.28) lies in the transverse space. We want to express this integral in terms of scalar integrals.

Note that in contrast to the three-point function considered in the preceeding Section, the rank-two two-point function in two dimensions has an ultra-viloet divergence. We regularize this divergence by continuing the loop momentum to $d = 2-2\epsilon$ dimensions and begin by constructing the van Neerven - Vermaseren basis. As the basis vector of the physical space, we take

$$n^\mu = \frac{k^\mu}{\sqrt{k^2}}, \quad n^2 = 1. \tag{5.29}$$

We choose $\hat{n}$ to be the basis vector of the transverse space, which is allowed since n and $\hat{n}$ are orthogonal, $n\cdot\hat{n} = 0$. As the consequence of the completeness relation, the two vectors satisfy

$$n^\mu n^\nu + \hat{n}^\mu \hat{n}^\nu = g^{\mu\nu}_{(2)}, \tag{5.30}$$

where $g^{\mu\nu}_{(2)}$ is the two-dimensional metric tensor. Contracting this equation with the loop momentum, we obtain

$$(\hat{n}\cdot l)^2 = l^2_{(2)} - (n\cdot l)^2 = l^2_{(2)} - \frac{(l\cdot k)^2}{k^2}. \tag{5.31}$$

Because l is a d-dimensional vector, we can decompose it as

$$l^\mu = (l\cdot n)n^\mu + (l\cdot\hat{n})\hat{n}^\mu + n^\mu_\epsilon (l\cdot n_\epsilon), \tag{5.32}$$

where n_ϵ is the unit vector that parametrizes the $(d-2)$-dimensional vector space. It follows that the square of the d-dimensional loop momentum can be written as

$$l^2 = l^2_{(2)} + (n_\epsilon\cdot l)^2 = l^2_{(2)} + \mu^2, \tag{5.33}$$

where $\mu^2 = (n_\epsilon\cdot l)^2$ is introduced. To proceed further, we express various scalar products through inverse Feynman propagators $d_{1,2}$

$$l^2_{(2)} = d_1 + m_1^2 - \mu^2, \quad 2l\cdot k = d_2 - d_1 - r_1^2, \tag{5.34}$$

and use Eqs. (5.31,5.33) to obtain

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = -\frac{(\lambda^2+\mu^2)}{d_1 d_2} + \frac{1}{4k^2}\left[\frac{r_1^2 - 2l\cdot k}{d_1} + \frac{r_2^2 + 2l\cdot k + 2k^2}{d_2}\right]. \tag{5.35}$$

In Eqs. (5.34,5.35), we use the following short-hand notations

$$\begin{aligned} r_1^2 &= k^2 + m_1^2 - m_2^2, \quad r_2^2 = k^2 + m_2^2 - m_1^2, \\ \lambda^2 &= \frac{k^4 - 2k^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4k^2}. \end{aligned} \tag{5.36}$$

Even if we did not know the result displayed in Eq. (5.35), we could still argue on general grounds that the integrand can be written as

$$\begin{aligned} \frac{(\hat{n} \cdot l)^2}{d_1 d_2} =& \frac{b_0 + b_1(\hat{n} \cdot l) + b_2(n_\epsilon \cdot l)^2}{d_1 d_2} + \frac{a_{1,0} + a_{1,1}(n \cdot l) + a_{1,2}(\hat{n} \cdot l)}{d_1} \\ &+ \frac{a_{2,0} + a_{2,1}(n \cdot l) + a_{2,2}(\hat{n} \cdot l)}{d_2}. \end{aligned} \tag{5.37}$$

We will explain in later where this parametrization comes from. Here, we compare terms in Eq. (5.35) and Eq. (5.37) and obtain

$$\begin{aligned} & b_0 = -\lambda^2, \ \ b_1 = 0, \ \ b_2 = -1, \\ & a_{1,0} = \frac{r_1^2}{4k^2}, \ \ a_{1,1} = -\frac{1}{2\sqrt{k^2}}, \ \ a_{1,2} = 0, \\ & a_{2,0} = \frac{r_2^2}{4k^2} + \frac{1}{2}, \ \ a_{2,1} = \frac{1}{2\sqrt{k^2}}, \ \ a_{2,2} = 0. \end{aligned} \tag{5.38}$$

It is instructive to rederive Eq. (5.38) using an alternative procedure. This procedure is important because it generalizes to four-dimensions, without modification, and because it shows how the reduction techniques are connected to unitarity. We begin by multiplying both sides of Eq. (5.37) by d_1, d_2 and obtain

$$\begin{aligned} (\hat{n} \cdot l)^2 = & \left[b_0 + b_1(\hat{n} \cdot l) + b_2(n_\epsilon \cdot l)^2\right] + \left[a_{1,0} + a_{1,1}(n \cdot l) + a_{1,2}(\hat{n} \cdot l)\right] d_2 \\ & + \left[a_{2,0} + a_{2,1}(n \cdot l) + a_{2,2}(\hat{n} \cdot l)\right] d_1. \end{aligned} \tag{5.39}$$

We would like to use Eq. (5.39) to find all the b- and a-coefficients. Since there are nine unknowns, we can evaluate Eq. (5.39) for nine different values of the loop momentum l, invert the nine-by-nine matrix and find the coefficients. While this procedure does, indeed, provide a solution to the problem, it involves inverting a large matrix and is therefore impractical. A better algorithm exploits the fact that, under special choices of the loop momentum l in Eq. (5.39), the matrix to invert becomes block-diagonal.

To see how this works, we first describe a procedure to compute the b-coefficients *only*. To project the right hand side of Eq. (5.39) onto b-coefficients, we choose the loop momentum l to satisfy $d_1(l) = d_2(l) = 0$. For the moment, consider the loop momentum l that satisfies those constraints and, simultaneously, has zero projection on the d-dimensional space, $n_\epsilon \cdot l = 0$. We find that there are just two loop momenta l that satisfy those constraints; they can be written as

$$l_c^\pm = \alpha_c n \pm i\beta_c \hat{n}, \tag{5.40}$$

where

$$\alpha_c = -\frac{r_1^2}{2\sqrt{k^2}}, \quad \beta_c = \lambda. \tag{5.41}$$

The parameters r_1 and λ are shown in Eq. (5.36). We substitute these two solutions into Eq. (5.39) and obtain two equations for the coefficients $b_{0,1}$

$$b_0 + b_1 \hat{n} \cdot l_c^+ = -\lambda^2, \quad b_0 + b_1 \hat{n} \cdot l_c^- = -\lambda^2. \tag{5.42}$$

It follows that $b_0 = -\lambda^2$ and $b_1 = 0$, in agreement with Eq. (5.38).

To find b_2 we proceed along similar lines but we require that the scalar product $l \cdot n_\epsilon$ does not vanish. Since the conditions $d_1 = 0, d_2 = 0$ are equivalent to $2l \cdot k + r_1^2 = 0$, $l^2 = m_1^2$, the loop momentum that satisfies those constraints is the same as in Eq. (5.40), up to a change $\hat{n} \to n_\epsilon$,

$$l^\pm = \alpha_c n \pm i\beta_c n_\epsilon. \tag{5.43}$$

Substituting $l^\pm$ into Eq. (5.39) and using $b_0 = -\lambda^2$, $b_1 = 0$, we obtain

$$0 = (1 + b_2)\lambda^2, \tag{5.44}$$

which implies that $b_2 = -1$, in agreement with Eq. (5.38).

The next step is to identify the coefficients of the tadpoles in Eq. (5.39). We will focus on a set $a_{1,0}, a_{1,1}, a_{1,2}$. We can project Eq. (5.39) on these coefficients by choosing the loop momentum for which d_1 vanishes but d_2 is different from zero. Note that no $l \cdot n_\epsilon$ terms are needed to find the a-coefficients. As the consequence, we can work with the two-dimensional loop momentum

$$l_1 = \gamma_1 n + \gamma_2 \hat{n}. \tag{5.45}$$

The equation $d_1(l_1) = 0$ implies $\gamma_1^2 + \gamma_2^2 = m_1^2$, so that γ_1, γ_2 lie on a circle of a radius m_1. Substituting l_1 into Eq. (5.39), we find

$$\gamma_2^2 = -\lambda^2 + (2\sqrt{k^2}\gamma_1 + r_1^2)(a_{1,0} + a_{1,1}\gamma_1 + a_{1,2}\gamma_2). \tag{5.46}$$

To solve Eq. (5.46), we choose $\gamma_1 = 0, \gamma_2 = \pm m_1$ and obtain two equations

$$a_{1,0} \pm a_{1,2} m_1 = \frac{m_1^2 + \lambda^2}{r_1^2} = \frac{r_1^2}{4k^2}. \tag{5.47}$$

Hence, it follows that $a_{1,0} = r_1^2/(4k^2)$ and $a_{1,2} = 0$, in agreement with Eq. (5.38). To find $a_{1,1}$, we choose $\gamma_2 = 0, \gamma_1 = m_1$, solve Eq. (5.46) and obtain $a_{1,1} = -(4k^2)^{-1/2}$.

We can determine coefficients $a_{2,0}, a_{2,1}, a_{2,2}$ in the same manner, by choosing the loop momentum that satisfies $d_2(l) = 0$. The calculation is similar to the one performed above and for this reason we do not present it here. We emphasize that the procedure that we just explained implies that, for the reduction of one-loop integrals to a set of scalar integrals, we need to know integrands at *special values of the loop momenta*, for which at least one of the inverse Feynman propagators that contributes to a particular diagram, vanishes. Since zeros of Feynman denominators correspond to situations when virtual particles go on their mass shells, the connection between the reduction procedure and the ideas of unitarity begins to emerge.

In this section, we generalize the two-dimensional reduction procedure described in the previous section to D-dimensional space-time. We are ultimately interested in the limit $D \to 4$.

5.3 Parametrization of the integrand

We begin with the observation that, in any renormalizable quantum field theory, the rank of the one-loop tensor integrals that appear does not exceed the number of external lines. Therefore, we will only be concerned with the reduction of one-loop integrals of restricted rank, e.g. the rank-five or less for five-point functions, rank-four or less for four-point functions and so on.

We would like to establish a simple parametrization of one-loop integrands, first introduced by Ossola, Papadopoulos and Pittau [17]. It reads

$$\begin{aligned} I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Bigg\{ & \\ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l) & \\ + \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l) & \\ + \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l) & \\ + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \Bigg\}. & \end{aligned} \tag{5.48}$$

The index i runs over all possible inverse Feynman propagators d_i. Similarly, the index j runs over all inverse Feynman propagators, except those explicitly excluded. The important feature of this parametrization is that all inverse propagators $d_i(l)$ on the right hand side appear in the first power, i.e. there are no terms of the form $d_i^2(l)$ for any i. In the spirit of the previous section, this allows us to project on different $\tilde{e}, \tilde{d}, \tilde{c}, \tilde{b}$ and $\tilde{a}$-functions, by considering loop momenta that nullify different sets of inverse propagators.

We will discuss first the reduction of a rank-five five-point function; the general case then easily follows. To this end, we consider $d_i(l) = (l+q_i)^2 - m_i^2$, $i = 0, \ldots, 4$, $q_0 = 0$ and assume that the numerator function reads

$$N(l) = \prod_{i=1}^{5} u_i \cdot l, \tag{5.49}$$

where u_i are some external four-dimensional vectors.

As the first step in the reduction procedure, we find the reduction coefficients of the five-point function, $\tilde{e}_{01234}$. To accomplish this, we construct the van Neerven-Vermaseren basis out of four vectors q_i and decompose the loop momentum

$$l^\mu = \sum_{i=1}^{4} (l \cdot q_i) v_i^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu. \tag{5.50}$$

The scalar products $l \cdot q_i$ are expressed in terms of inverse Feynman propagators

$$l \cdot q_i = \frac{1}{2} \left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2) \right). \tag{5.51}$$

Since $u_5 \cdot n_\epsilon = 0$, we can rewrite Eq. (5.49) as

$$
\begin{aligned}
N(l) = \left(\prod_i^4 u_i \cdot l\right)(u_5 \cdot l) = \frac{1}{2}\sum_{j=1}^4 (u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(d_j - d_0) \\
- \frac{1}{2}\sum_{j=1}^4 (u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2).
\end{aligned}
\tag{5.52}
$$

Upon dividing the numerator function by the product of inverse Feynman propagators $d_0 d_1 d_2 d_3 d_4$, we find that the first term on the right-hand-side of Eq. (5.52), produces a collection of rank-four four-point functions and the second term – a rank-four five-point function. We now repeat the same procedure with the rank-four five-point function and conclude that it can be expressed through a combination of rank-three four-point functions and the rank-three five point function. Whenever, as a result of these manipulations, the propagator d_0 cancels, it is possible to shift the loop-momentum to bring the integrand to the standard form. We can clearly continue this procedure until we are left with a *scalar* five-point function and a collection of four-point functions of the ranks from zero (scalar) to four (maximal). Hence, we have established that the function $\tilde{e}_{01234}(l)$ in Eq. (5.48) is l-independent

$$
\tilde{e}_{01234}(l) = e_0. \tag{5.53}
$$

In the course of the procedure described above, the highest rank integral left unreduced is the rank-four four-point function. We now discuss how it can be reduced. For definiteness, we consider the four-point function with four propagators d_0, d_1, d_2, d_3, but our discussion can be applied to any other four-point function, by the appropriate re-definition of the propagator momenta and masses. We construct van Neerven-Vermaseren basis vectors out of the three momenta q_1, q_2, q_3. The physical space in this case is three-dimensional and the transverse space is one-dimensional. We parametrize the transverse space by the unit vector n_4.

The decomposition in terms of van Neerven-Vermaseren basis then reads

$$
l^\mu = \sum_{i=1}^3 v_i^\mu (l \cdot q_i) + (l \cdot n_4) n_4^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu. \tag{5.54}
$$

Using this parametrization we can write

$$
\begin{aligned}
N_4(l) = \left(\prod_i^3 u_i \cdot l\right)(u_4 \cdot l) = \frac{1}{2}\sum_{j=1}^3 (u_4 \cdot v_j)\left(\prod_i^3 u_i \cdot l\right)(d_j - d_0) \\
- \frac{1}{2}\sum_{j=1}^3 (u_4 \cdot v_j)\left(\prod_i^3 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2) + \left(\prod_i^3 u_i \cdot l\right)(lcdotn_4)(u_4 \cdot n_4).
\end{aligned}
\tag{5.55}
$$

The first two terms on the right-hand side are considered "reduced", since they are rank-three three-point and four-point functions. The last term, however, is a rank-four four-point function, and so it does not appear that we gained anything. To demonstrate that we, actually, did gain something, we take the last term in Eq. (5.55) and repeat the reduction procedure described above. It is clear that a variety of terms will be produced, most of lower-point or lower-rank type, and the only term that we should consider as "not-reduced" reads

$$\left(\prod_i^3 u_i \cdot l\right)(l \cdot n_4) \to \left(\prod_i^2 u_i \cdot l\right)(l \cdot n_4)^2. \tag{5.56}$$

We simplify it by examining the square of the loop momentum l. Using the decomposition in terms of van Neerven-Vermaseren basis, eq. (5.54) and the relations $2l \cdot q_i = (d_i - d_0 - q_i^2 + m_i^2 - m_0^2)$ and $l^2 = d_0 + m_0^2$, we find

$$(l \cdot n_4)^2 = -(l \cdot n_\epsilon)^2 + \text{constant terms} + \mathcal{O}(d_0, d_1, d_2, d_3). \tag{5.57}$$

Terms dubbed "constant" in the above formula contribute (after multiplication by $(u_1 \cdot l)(u_2 \cdot l)$) to rank-two four-point functions while terms that contain at least one inverse Feynman propagator, contribute to three-point functions. The "not-reduced" part of the rank-four four-point function therefore reads

$$\prod_i^4 u_i \cdot l \to \left(\prod_i^2 u_i \cdot l\right)(l \cdot n_4)^2 \to \left(\prod_i^2 u_i \cdot l\right)(l \cdot n_\epsilon)^2. \tag{5.58}$$

It is clear that if we repeat the reduction process, we express any tensor four-point function integral (of rank not higher than four), through the following numerator function

$$\tilde{d}_{0123}(l) = \tilde{d}_0 + \tilde{d}_1(l \cdot n_4) + \tilde{d}_2(l \cdot n_\epsilon)^2 + \tilde{d}_3(l \cdot n_\epsilon)^2(l \cdot n_4) + \tilde{d}_4(l \cdot n_e)^4, \tag{5.59}$$

where the l-dependence is shown explicitly. We note that the degree of the polynomial on the right hand side of Eq. (5.59) is the direct consequence of the fact that the highest rank tensor four-point functions that we consider is four. This restriction works well if we deal with renormalizable quantum field theories but it might not be general enough if one-loop calculations with effective field theories are attempted. The extension of the algorithm to those cases is straightforward since the required parametrization of a numerator function of, say, a four-point function will be an extension of Eq. (5.59) to higher rank tensors. It is straightforward to figure out the required extension, following the line of reasoning explained above. Interestingly, such extensions are very economical; for example, we mention that to achieve a reduction of a *rank-five* four-point functions, we only need to include one additional term $\tilde{d}_5(l \cdot n_\epsilon)^4(l \cdot n_4)$ in the parametrization of $\tilde{d}_{0123}$ in Eq. (5.59).

We now turn our attention to the three-point functions that are obtained in the course of the reduction of the four-point functions. The highest tensor rank we have to care about is three. The physical space is two-dimensional and the transverse space is two-dimensional as well. The loop momentum reads

$$l^\mu = \sum_{i=1}^2 v_i^\mu (l \cdot q_i) + (l \cdot n_3)n_3^\mu + (l \cdot n_4)n_4^\mu + (l \cdot n_\epsilon)n_\epsilon^\mu. \tag{5.60}$$

We follow the same procedure as already described in the context of five- and four-point functions. The reduced terms will be at most rank-two two-point functions. The irreducible structures read

$$\prod_{i=3}^{4}(l\cdot u_i) \to \sum_{i=3}^{4} c_{1i}(l\cdot n_i) + \sum_{i=3}^{4} c_{2i}(l\cdot n_i)^2 + \sum_{i=3}^{4} c_{3i}(l\cdot n_i)^3 \\ + c_4(l\cdot n_4)(l\cdot n_3) + c_5(l\cdot n_3)^2(l\cdot n_4) + c_6(l\cdot n_3)(l\cdot n_4)^2 \tag{5.61}$$

Similar to the case of the four-point function, not all the terms in Eq. (5.61) are independent in the four-dimensional case. To make this dependence explicit, we square Eq. (5.60), use $l^2 = d_0 + m_0^2$ and find

$$(l\cdot n_3)^2 + (l\cdot n_4)^2 + (l\cdot n_\epsilon)^2 = \text{constant terms} + \mathcal{O}(d_0, d_1, d_2). \tag{5.62}$$

We use this constraint in Eq. (5.61), to trade $(l\cdot n_3)^2(l\cdot n_4)$, $(l\cdot n_4)^2(l\cdot n_3)$ for $(l\cdot n_\epsilon)^2(l\cdot n_4)$ and $(l\cdot n_\epsilon)^2(l\cdot n_3)$. Also, given Eq. (5.62), we can use $(l\cdot n_\epsilon)^2$ and $(l\cdot n_3)^2 - (l\cdot n_4)^2$ as two independent structures, instead of $(l\cdot n_3)^2$ and $(l\cdot n_4)^2$. Hence, the parametrization of the function $\tilde{c}_{012}$ becomes

$$\tilde{c}_{012}(l) = \tilde{c}_0 + \tilde{c}_1(l\cdot n_3) + \tilde{c}_2(l\cdot n_4) + \tilde{c}_3((l\cdot n_3)^2 - (l\cdot n_4)^2) \\ + \tilde{c}_4(l\cdot n_3)(l\cdot n_4) + \tilde{c}_5(l\cdot n_3)^3 + \tilde{c}_6(l\cdot n_4)^3 \\ + \tilde{c}_7(l\cdot n_\epsilon)^2 + \tilde{c}_8(l\cdot n_\epsilon)^2(l\cdot n_3) + \tilde{c}_9(l\cdot n_\epsilon)^2(l\cdot n_4). \tag{5.63}$$

The advantage of this parametrization, compared to Eq. (5.60), is that in four dimensions only $\tilde{c}_0$ gives a non-vanishing contribution after integration.

Similar considerations can be used to derive the general parametrization of the two-point and one-point functions. Recall that the highest tensor rank of the two-point function that we consider is two; the highest tensor rank of the one-point function is one. We will not discuss the derivation and just give the results for the numerator functions. The numerator of the two-point function can be written as

$$\begin{aligned} \tilde{b}_{01}(l) &= \tilde{b}_0 + \tilde{b}_1(l\cdot n_2) + \tilde{b}_2(l\cdot n_3) + \tilde{b}_3(l\cdot n_4) \\ &+ \tilde{b}_4((l\cdot n_2)^2 - (l\cdot n_4)^2) + \tilde{b}_5((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{b}_6(l\cdot n_2)(l\cdot n_3) \\ &+ \tilde{b}_7(l\cdot n_3)(l\cdot n_4) + \tilde{b}_8(l\cdot n_2)(l\cdot n_4) + \tilde{b}_9(l\cdot n_\epsilon)^2, \end{aligned} \tag{5.64}$$

while the general parametrization of the numerator of the one-point function for propagator d_i is

$$\tilde{a}_i(l) = \tilde{a}_0 + \tilde{a}_1(l\cdot n_1) + \tilde{a}_2(l\cdot n_2) + \tilde{a}_3(l\cdot n_3) + \tilde{a}_4(l\cdot n_4). \tag{5.65}$$

In Equation (5.65) $\tilde{a}_0$ is the relevant reduction coefficient since all other terms integrate to zero.

5.4 How to compute the reduction coefficients

In the previous Section we showed how an integrand of a general one-loop integral in a renormalizable quantum field theory can be parametrized. An important feature of this parametrization is that all l-dependent four-dimensional tensors that are present in the coefficients $\tilde{e}_{i_1..i_5}, .., \tilde{a}_{i_1}$ are evanescent under angular integration in the transverse space of the respective reduced integral. We will refer to such tensors as "traceless". This feature is extremely important since it allows us to perform the integration immediately and rewrite Eq. (5.48) in a simplified, fully reduced form

$$
\begin{aligned}
I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} &= \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}^{(0)}_{i_1,i_2,i_3,i_4,i_5} I_{i_1 i_2 i_3 i_4 i_5} \\
&+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}^{(0)}_{i_1,i_2,i_3,i_4} I_{i_1 i_2 i_3 i_4} + \sum_{i_1,i_2,i_3} \tilde{c}^{(0)}_{i_1,i_2,i_3} I_{i_1 i_2 i_3} \\
&+ \sum_{i_1,i_2} \tilde{b}^{(0)}_{i_1,i_2} I_{i_1 i_2} + \sum_{i_1} \tilde{a}^{(0)}_{i_1} I_{i_1} + \mathcal{R}.
\end{aligned}
\tag{5.66}
$$

The right hand side of Eq. (5.66) contains scalar integrals multiplied by l-independent contributions of the reduction coefficients $\tilde{e}^{(0)}, \tilde{d}^{(0)}, \tilde{b}^{(0)}$, etc. and the "rational" term $\mathcal{R}$ which originates from the integration over the loop momentum of tensorial structures involving $(l \cdot n_\epsilon)$. Eq. (5.66) gives an explicit demonstration of the reduction formula stated in Eq. (**??**).

The reason the integration over the loop momentum is so simple is because the projection on the transverse space is always given in terms of traceless tensors. To illustrate this point, consider a contribution of a general two-point function to the right-hand side of Eq. (5.48). It reads

$$
\begin{aligned}
&\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{(l^2 - m_0^2)(l^2 + 2l\cdot q + q^2 - m_1^2)} \Big\{ \tilde{b}_0 + \tilde{b}_1 (l \cdot n_2) \\
&+ \tilde{b}_2 (l \cdot n_3) + \tilde{b}_3 (l \cdot n_4) + \tilde{b}_4 ((l \cdot n_2)^2 - (l \cdot n_4)^2) \\
&+ \tilde{b}_5 ((l \cdot n_3)^2 - (l \cdot n_4)^2) + \tilde{b}_6 (l \cdot n_2)(l \cdot n_3) + \tilde{b}_7 (l \cdot n_3)(l \cdot n_4) \\
&+ \tilde{b}_8 (l \cdot n_2)(l \cdot n_4) + \tilde{b}_9 (l \cdot n_\epsilon)^2 \Big\}.
\end{aligned}
\tag{5.67}
$$

Because $q \cdot n_\epsilon = 0, q \cdot n_i = 0$, $i = 2,3,4$, the integration over the directions of the transverse space $l_\perp = n_2(l\cdot n_2) + n_3(l\cdot n_3) + n_4(l \cdot n_4) + n_\epsilon(l \cdot n_\epsilon)$ is straightforward. We obtain

$$
\int \mathrm{d}^{D-1} l_\perp \delta(l_\perp^2 - \mu_0^2) \left(l_\perp^\mu, l_\perp^\mu l_\perp^\nu \right) = \int \mathrm{d}^{D-1} l_\perp \delta(l_\perp^2 - \mu_0^2) \left(0, \frac{g_\perp^{\mu\nu}}{D-1} \, l_\perp^2 \right). \tag{5.68}
$$

Using this result in Eq. (5.67) together with the orthonormality property of the transverse space basis vectors $n_i n_j = \delta_{ij}$, we conclude that only two terms – $\tilde{b}_0$ and $\tilde{b}_9$ contribute after the integration over the loop momentum is performed. The term with $\tilde{b}_9$ contributes to the rational part $\mathcal{R}$ in Eq. (5.66), while $\tilde{b}_0$ is the reduction coefficient of the relevant two-point master integral. Similar considerations apply to all other reduction coefficients, leading to Eq. (5.66). Clearly, we are interested in the calculation of quantities that are integrated over the loop momentum. It follows from Eq. (5.66) that, in addition to the rational part, we only require a modest number of the reduction coefficients $\tilde{e}^{(0)}, \tilde{d}^{(0)}, \tilde{c}^{(0)}, ..$ etc. The question that we address now is how to find those coefficients efficiently.

In the course of the discussion of the two-dimensional case, we have seen that a powerful way to find coefficients $\tilde{e}_{i_1..i_5}, ..., \tilde{a}_{i_1}$ involves calculations of both sides of Eq. (5.48) for special values of the loop momentum l, where a chosen subset of inverse Feynman propagators $d_1, d_2, ..., d_N$ vanish. We now discuss this procedure in detail, pointing out some subtleties that appear once we implement it.

We begin with the five-point function contribution. We choose five inverse propagators, say $d_0, d_1, ...d_4$ and find the loop momentum for which *all* of these inverse propagators vanish. This requires the momentum l to span more than four dimensions, so, for definiteness, we make the minimal choice and take l to be five-dimensional. Clearly, the only term in the right hand side of Eq. (5.48) that is non-zero is the term that does not contain any of the five propagators. This is $\tilde{e}_{01234}$ – the term that we would like to find. We argued previously that this term is constant, so computing the left hand side of Eq. (5.48) with the momentum l^* such that $d_0(l^*) = 0, d_1(l^*) = 0, ...d_4(l^*) = 0$, gives us $\tilde{e}_{01234}$.

While this procedure is correct, it often becomes impractical since it treats the scalar five-point function as a master integral. This would have been fully justified if we were interested in a five-dimensional calculation, but, in practical computations, we eventually take the limit $D \to 4$. In this limit, the five-point function becomes a linear combination of five four-point functions. We would therefore like to eliminate the five-point integral from the set of master integrals right away, avoiding large cancellations between four- and five-point functions in the $D \to 4$ limit. To see how this can be done note that the loop momentum in the five-point function can be written as

$$l^\mu = \frac{1}{2}\sum_{i=1}^{4} v_i^\mu \left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right) + (l \cdot n_\epsilon) n_\epsilon^\mu. \tag{5.69}$$

Squaring the two sides of this equation and using $l^2 = d_0 + m_0^2$, we see that for a loop momentum that satisfies $d_0 = 0, d_1 = 0, \ldots d_5 = 0$, we have

$$(l \cdot n_\epsilon)^2 = -\frac{1}{4}\sum_{ij}(v_i \cdot v_j)(q_i^2 - m_i^2 + m_0^2)(q_j^2 - m_j^2 + m_0^2) + m_0^2\,. \tag{5.70}$$

It follows that we can either choose a scalar five-point function as the master integral or the integral with additional $(l \cdot n_\epsilon)^2$ in the numerator. However, because

$$\lim_{D\to 4}\int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{(l \cdot n_\epsilon)^2}{d_1 d_2 d_3 d_4 d_5} \to 0, \tag{5.71}$$

the second choice is preferable. Indeed, since the new master integral that we introduced to account for the need to employ dimensional regularization does not contribute in the $D \to 4$ limit, all four-dimensional relations between various integrals are automatically accounted for. Therefore this coefficient is only needed as a subtraction term in the determination of lower point coefficients. Experience shows that adopting this alternative definition of the pentagon coefficient leads to improved numerical stability in practical computations[18, 19].

To find the other coefficients, we follow the strategy already discussed in the context of two-dimensional computations. For example, having determined the five-point functions, we subtract their coefficients from the left-hand side of Eq. (5.48) and consider all the subsets of four propagators. We focus on one subset, $d_0, ..., d_3$ whose contribution is described by the coefficient

$$\tilde{d}_{0123} = \tilde{d}_0 + \tilde{d}_1(l \cdot n_4) + \tilde{d}_2(l \cdot n_\epsilon)^2 + \tilde{d}_3(l \cdot n_\epsilon)^2(l \cdot n_4) + \tilde{d}_4(l \cdot n_e)^4. \tag{5.72}$$

To determine $\tilde{d}_{0123}$, we find a momentum l that satisfies $d_0(l) = 0$, $d_1(l) = 0$, $d_2(l) = 0$, $d_3(l) = 0$ and write it as

$$\begin{aligned} l^\mu &= V^\mu + l_\perp(\cos\phi\, n_4^\mu + \sin\phi\, n_\epsilon^\mu), \\ V^\mu &= -\frac{1}{2}\sum_i^3 v_i^\mu \left(q_i^2 - m_i^2 + m_0^2\right), \end{aligned} \tag{5.73}$$

with $l_\perp = \sqrt{l_\perp \cdot l_\perp}$. Again it is sufficient to consider l to be five dimensional. The length of the projection of the vector l on the transverse space is fixed

$$l_\perp^2 = m_0^2 - V_\mu V^\mu. \tag{5.74}$$

To find the $\tilde{d}_0, ...\tilde{d}_4$ coefficients, we take for instance $\sin\phi = 0, \cos\phi = \pm 1$, denote $l_\pm^\mu = V^\mu \pm l_\perp n_4^\mu$, calculate the numerator for these values of the loop momenta and find

$$\tilde{d}_0 = \frac{\mathrm{Num}(l_+) + \mathrm{Num}(l_-)}{2}, \quad \tilde{d}_1 = \frac{\mathrm{Num}(l_+) - \mathrm{Num}(l_-)}{2l_\perp}. \tag{5.75}$$

To find $\tilde{d}_{2,3,4}$, we need to do a little bit more. First, we take $\cos\phi = \sin\phi = \pm 1/\sqrt{2}$, denote the loop momentum as $\tilde{l}_\pm = V \pm l_\perp(n_4 + n_\epsilon)/\sqrt{2}$, and find

$$\begin{aligned} \tilde{d}_2 + \tilde{d}_4\frac{l_\perp^2}{2} &= \frac{1}{l_\perp^2}\left(\mathrm{Num}(\tilde{l}_+) + \mathrm{Num}(\tilde{l}_-) - 2\,\tilde{d}_0\right), \\ \tilde{d}_3 &= \frac{\sqrt{2}}{l_\perp^3}\left(\mathrm{Num}(\tilde{l}_+) - \mathrm{Num}(\tilde{l}_-) - \sqrt{2}\tilde{d}_1 l_\perp\right). \end{aligned} \tag{5.76}$$

We need yet another equation to resolve the $\tilde{d}_2 - \tilde{d}_4$ degeneracy. It is convenient to take $l_\epsilon^\mu = V^\mu + l_\perp n_\epsilon^\mu$; this leads to

$$\tilde{d}_2 + \tilde{d}_4 l_\perp^2 = \frac{\mathrm{Num}(l_\epsilon) - \tilde{d}_0}{l_\perp^2}. \tag{5.77}$$

We find

$$\begin{aligned} \tilde{d}_2 &= \frac{1}{l_\perp^2}\left(2\mathrm{Num}(\tilde{l}_+) + 2\mathrm{Num}(\tilde{l}_-) - \mathrm{Num}(l_\epsilon) - 3\tilde{d}_0\right), \\ \tilde{d}_4 &= \frac{2}{l_\perp^4}\left(\mathrm{Num}(l_\epsilon) - \mathrm{Num}(\tilde{l}_+) - \mathrm{Num}(\tilde{l}_-) + \tilde{d}_0\right). \end{aligned} \tag{5.78}$$

We next discuss how to compute the coefficients of the three-point functions. As an illustration, we choose a three point function with denominators d_0, d_1, d_2; its contribution is described by a coefficient

$$\begin{aligned} \tilde{c}_{012} = \tilde{c}_0 &+ \tilde{c}_1(l \cdot n_3) + \tilde{c}_2(l \cdot n_4) + \tilde{c}_3((l \cdot n_3)^2 - (l \cdot n_4)^2) \\ &+ \tilde{c}_4(l \cdot n_3)(l \cdot n_4) + \tilde{c}_5(l \cdot n_3)^3 + \tilde{c}_6(l \cdot n_4)^3 \\ &+ \tilde{c}_7(l \cdot n_\epsilon)^2 + \tilde{c}_8(l \cdot n_\epsilon)^2(l \cdot n_3) + \tilde{c}_9(l \cdot n_\epsilon)^2(l \cdot n_4). \end{aligned} \tag{5.79}$$

We choose the loop momentum that satisfies $d_0(l) = d_1(l) = d_2(l) = 0$ and parametrize it as

$$l^\mu = V^\mu + l_\perp(x_3 n_3^\mu + x_4 n_4^\mu + x_\epsilon n_\epsilon^\mu). \tag{5.80}$$

Consider the class of momenta with $x_\epsilon = 0$; such a choice allows us to determine the coefficients $\tilde{c}_{0,..6}$. If $x_\epsilon = 0$, $x_3^2 + x_4^2 = 1$, so that we can take $x_3 = \cos\phi, x_4 = \sin\phi$. It is convenient then to rewrite Eq. (5.79) as a polynomial in $t = e^{i\phi}$. Equation (5.79) becomes

$$\tilde{c}_{012}(t) = \sum_{k=-3}^{3} c_k t^k, \tag{5.81}$$

where the coefficients c_k read

$$\begin{aligned} c_{\pm3} &= \frac{\tilde{c}_5 \pm i\tilde{c}_6}{8} l_\perp^3, \quad c_{\pm2} = \frac{2\tilde{c}_3 \mp i\tilde{c}_4}{4} l_\perp^2, \\ c_{\pm1} &= \left(\frac{1}{2}\,\tilde{c}_1 \mp \frac{i}{2}\,\tilde{c}_2,\right) l_\perp + \left(\frac{3}{8}\,\tilde{c}_5 \mp \frac{3i}{8}\,\tilde{c}_6\right) l_\perp^3\,, \end{aligned} \tag{5.82}$$

and $c_0 = \tilde{c}_0$. We can now use the technique of discrete Fourier transform, first discussed in the context of the OPP reduction, in Refs. [20, 21]. Application of the discrete Fourier transform allows us to write explicit expressions for the coefficients c_k in a straightforward way. Indeed, they are given by

$$c_m = \frac{1}{7}\sum_{n=0}^{6} \tilde{c}_{012}(t_n)\, t_n^{-m}, \tag{5.83}$$

where $t_n = e^{2\pi\, i\, n/7}$. To prove this equation, note that

$$\sum_{n=0}^{k} e^{\frac{2\pi i n}{k+1} r} = \delta_{r0}(k+1). \tag{5.84}$$

Hence, substituting Eqs. (5.81) into the right hand side of Eq. (5.83) and carrying out the summation over n using Eq. (5.84), we can easily show that the right hand side of Eq. (5.83) is indeed one of the c-coefficients. Hence, Eq. (5.83) provides a convenient way to find the cut-constructible coefficients of the three-point function. Finally, to determine the rational part coefficients in Eq. (5.79), we take vectors l that have non-vanishing projections on either n_3 and n_ϵ or on n_4 and n_ϵ. Since we already know all the cut-constructible coefficients, it is straightforward to find $\tilde{c}_{7,8,9}$.

We note that the discrete Fourier transform is just one of many ways to solve the linear system of equations required to obtain the coefficients $\tilde{c}_0, ...\tilde{c}_6$. It is a convenient, easy-to-code-up procedure, but it is neither unique nor superior to other ways. In fact, it is clear that in certain cases it is better to avoid using the discrete Fourier transform method and to solve the system of equations by other means.

To see why this might be the case, we discuss the computation of the reduction coefficients for the two-point function with two propagators d_0 and d_1. Then, the physical space is one-dimensional and the transverse space is three-dimensional. The momentum parametrization therefore reads

$$l^\mu = x_1 q_1^\mu + l_\perp\left(\sum_{i=2}^{4} x_i n_i^\mu + x_\epsilon n_\epsilon^\mu\right), \quad q_1 \cdot n_{i\geq2} = 0, \quad n_i \cdot n_j = \delta_{ij}. \tag{5.85}$$

Using Eq. (5.85), we find that components of the momentum l for which $d_{1,2} = 0$ are subject to the following constraints

$$x_1 = \frac{(m_1^2 - m_0^2 - q_1^2)}{2q_1^2}, \quad l_\perp^2 = m_0^2 - x_1^2 q_1^2, \quad x_2^2 + x_3^2 + x_4^2 + x_\epsilon^2 = 1. \tag{5.86}$$

The general parametrization of the $\tilde{b}$-coefficient reads

$$\begin{aligned}
\tilde{b}_{01} &= \tilde{b}_0 + \tilde{b}_1(l\cdot n_2) + \tilde{b}_2(l\cdot n_3) + \tilde{b}_3(l\cdot n_4) \\
&+ \tilde{b}_4((l\cdot n_2)^2 - (l\cdot n_4)^2) + \tilde{b}_5((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{b}_6(l\cdot n_2)(l\cdot n_3) \\
&+ \tilde{b}_7(l\cdot n_3)(l\cdot n_4) + \tilde{b}_8(l\cdot n_2)(l\cdot n_4) + \tilde{b}_9(l\cdot n_\epsilon)^2.
\end{aligned} \tag{5.87}$$

Similar to the case of the three-point function, there are infinitely many loop momenta that satisfy the constraints shown in Eq. (5.86). Therefore, to find the cut-constructible coefficients, we can proceed as before, parametrizing

$$l_\perp^\mu = l_\perp\left(\sin\theta\cos\phi\, n_2^\mu + \sin\theta\sin\phi\, n_3^\mu + \cos\theta\, n_4^\mu\right), \tag{5.88}$$

and then applying the technique of the discrete Fourier transform to determine $\tilde{b}_0, \ldots \tilde{b}_8$. Note, however, that the application of the discrete Fourier transform requires division by $l_\perp$, c.f. Eq. (5.82) and this may lead to potential trouble. Indeed, according to Eq. (5.86), $l_\perp$ vanishes if $m_0^2 = x_1^2 q_1^2$ which corresponds to $q_1^2 = (m_0 - m_1)^2$ or $q_1^2 = (m_0 + m_1)^2$. These kinematic points are not dangerous if only massless virtual particles are considered. However, the situation may become problematic if virtual massive particles are present in the calculation. Note also that close to those exceptional values of q_1^2, $l_\perp$ can be small, so that division by $l_\perp$ may lead to numerical instabilities.

To handle the case of small $l_\perp$ in a numerically stable way, the method of discrete Fourier transform is not directly applicable and the system of equations must be solved differently. There are many ways to solve a system of linear equations avoiding division by $l_\perp$; one option is described below. We begin by choosing $l_\perp^\pm = x_\perp n_2 \pm x_3 n_3$, $l_\perp^\pm \cdot l_\perp^\pm = l_\perp^2$. Recall that $l_\perp^2$ is fixed by the on-shell condition Eq. (5.86) and therefore x_3 is expressed through $x_\perp$, $x_3 = \sqrt{l_\perp^2 - x_\perp^2}$. We calculate $b_\pm = b(l^\pm)$ and eliminate x_3^2 in favor of $l_\perp^2$ and $x_\perp$ where possible. We obtain

$$b_\pm = \tilde{b}_0 + \tilde{b}_1 x_\perp \pm x_3 \tilde{b}_2 + \tilde{b}_4 x_\perp^2 + \tilde{b}_5 x_3^2 \pm \tilde{b}_6 x_\perp x_3. \tag{5.89}$$

Taking the sum and the difference of $b_\pm$, we arrive at

$$\frac{(b_+ + b_-)}{2} = \tilde{b}_0^{\text{eff}} + \tilde{b}_1 x_\perp + \tilde{b}_4^{\text{eff}} x_\perp^2, \quad \frac{(b_+ - b_-)}{2x_3} = \tilde{b}_2 + \tilde{b}_6 x_\perp, \tag{5.90}$$

where

$$\begin{aligned}
\tilde{b}_0^{\text{eff}} &= \tilde{b}_0 + \tilde{b}_5 l_\perp^2, \\
\tilde{b}_4^{\text{eff}} &= \tilde{b}_4 - \tilde{b}_5.
\end{aligned} \tag{5.91}$$

The right hand sides of these equations are polynomials in $x_\perp$. Therefore, we can apply a discrete Fourier transform with respect to $x_\perp$ to find coefficients $\tilde{b}_1, \tilde{b}_4^{\text{eff}}, \tilde{b}_0^{\text{eff}}$ as well as $\tilde{b}_2, \tilde{b}_6$ in Eq. (5.90).

To determine the remaining coefficients, we make five choices of the loop-momentum, satisfying the on-shell condition. We choose for instance

$$\begin{aligned} l^{(a)} &= x_1 q_1 + x n_2 + y n_4\,, \\ l^{(b)} &= x_1 q_1 - x n_2 + y n_4\,, \\ l^{(c)} &= x_1 q_1 - x n_2 - y n_4\,, \\ l^{(d)} &= x_1 q_1 + x n_4 + y n_3\,, \\ l^{(e)} &= x_1 q_1 + x n_2 + y n_\epsilon\,. \end{aligned} \tag{5.92}$$

where $x^2 + y^2 = l_\perp^2$. We use the notation $b_\alpha = b(l^{(\alpha)})$. With the coefficients $\tilde{b}_0^{\rm eff}, \tilde{b}_1, \tilde{b}_2, \tilde{b}_4^{\rm eff}$ and $\tilde{b}_6$ in hand, we determine the other coefficients in the sequence, $\tilde{b}_8, \tilde{b}_3, \tilde{b}_5, \tilde{b}_7, \tilde{b}_9, \tilde{b}_0, \tilde{b}_4$. The results are

$$\begin{aligned} \tilde{b}_8 &= \frac{(\frac{1}{2}(b_a - b_b) - x\tilde{b}_1)}{xy}\,, \\ \tilde{b}_3 &= \frac{\frac{1}{2}(b_a - b_c) - \tilde{b}_1 x}{y}\,, \\ \tilde{b}_5 &= \frac{\tilde{b}_0^{\rm eff} + \tilde{b}_3 y + yx\tilde{b}_8 + x\tilde{b}_1 + (x^2 - y^2)\tilde{b}_4^{\rm eff} - b_a}{3y^2}\,, \\ \tilde{b}_7 &= \frac{(b_d - y^2\tilde{b}_5 + \tilde{b}_5 x^2 + \tilde{b}_4 x^2 - \tilde{b}_3 x - y\tilde{b}_2 - \tilde{b}_0)}{xy}\,, \\ \tilde{b}_9 &= \frac{(b_e - \tilde{b}_4 x^2 - \tilde{b}_1 x - \tilde{b}_0}{y^2}\,. \end{aligned} \tag{5.93}$$

The coefficients $\tilde{b}_0$ and $\tilde{b}_4$ are determined using Eq. (5.91) once $\tilde{b}_5$ has been fixed.

We have just described a method to calculate coefficients $\tilde{b}_{1,\dots 9}$ in a numerically stable way for small values of $l_\perp$. Note that we used the fact that even for arbitrarily small $l_\perp^2$ we can choose large, complex values of x, y with $x^2 + y^2 = l_\perp^2$. In the numerical program, we switch from the discrete Fourier transform to the solution just described, depending on the value of $l_\perp$. However, the described methods can only work *if* the decomposition of the loop momentum, as in Eq. (5.85), exists. A glance at Eq. (5.86) makes it clear that the decomposition fails for the *light-like* momentum, $q_1^2 = 0$, and we have to handle this case differently. We describe a possible solution below.

First, some clarifications are in order. Because we are interested in one-loop calculations for infra-red safe observables, it is reasonable to assume that the vector q_1 can be *exactly* light-like but it is impossible for that vector to be *nearly* light-like, since such kinematic configurations are, typically, rejected by cuts[2]. Hence, we have to modify the above analysis to allow for an exactly light-like external momentum. To this end, we choose a frame where the four-vector in Eq. (5.85) reads $q_1 = (E, 0, 0, E)$. We introduce a complementary light-like vector $\bar{q}_1 = (E, 0, 0, -E)$. The loop momentum is parametrized as $l = x_1 q_1 + x_2 \bar{q}_1 + l_\perp$. We denote the basis vectors of the transverse space as $n_{3,4}$;

[2]External particles with small masses are obvious exceptions but rarely do we need to know observables for, say, massive b-quarks in a situation when all kinematic invariants are large.

they satisfy $n_i n_j = \delta_{ij}$, $q_1 \cdot n_{3,4} = 0$, $\bar{q}_1 \cdot n_{3,4} = 0$. The on-shell condition for the loop momentum fixes x_2

$$x_2 = \frac{m_1^2 - m_0^2}{s}, \quad s = 2q_1\bar{q}_1, \tag{5.94}$$

and a linear combination of x_1 and $l_\perp^2$

$$l_\perp^2 + m_1^2 x_1 - m_0^2(1 + x_1) = 0. \tag{5.95}$$

Compared to the case when the reference vector q_1 is not on the light-cone, we write now the parametrization of the function $\tilde{b}$ using $n_4 \cdot l$. We choose it to be

$$\begin{aligned}\tilde{b}(l) = \tilde{b}_0 &+ \tilde{b}_1(\bar{q}_1 \cdot l) + \tilde{b}_2(n_3 \cdot l) + \tilde{b}_3(n_4 \cdot l) + \tilde{b}_4(\bar{q}_1 \cdot l)(\bar{q}_1 \cdot l) \\ &+ \tilde{b}_5(\bar{q}_1 \cdot l)(n_3 l) + \tilde{b}_6(\bar{q}_1 \cdot l)(n_4 \cdot l) + \tilde{b}_7((n_3 \cdot l)^2 - (n_4 \cdot l)^2) \\ &+ \tilde{b}_8(n_3 \cdot l)(n_4 \cdot l) + \tilde{b}_9(l \cdot n_\epsilon)^2.\end{aligned} \tag{5.96}$$

We describe a procedure to find the coefficients $\tilde{b}_0, \ldots \tilde{b}_9$ in a numerically stable way. To this end, we choose $x_1 = 0.5$. This fixes $l_\perp^2$, and x_2 is fixed by the on-shell condition Eq. (5.94). The freedom remains to choose the *direction* of the vector $l_\perp$ in the (n_3, n_4) plane. Consider four different vectors

$$l_\perp^{(a)} = yn_3 + xn_4,\ l_\perp^{(b)} = -yn_3 + xn_4,\ l_\perp^{(c)} = yn_3 - xn_4,\ l_\perp^{(d)} = -yn_3 - xn_4, \tag{5.97}$$

where $x^2 + y^2 = l_\perp^2$. We use vectors $l^{(\alpha)} = x_1 q_1 + x_2 \bar{q}_1 + l_\perp^{(\alpha)}$, $\alpha = a, b, c, d$, to calculate the function $b^{(\alpha)} = \tilde{b}(l^\alpha)$. Using $b_a, ... b_d$, we can immediately find the coefficient b_8

$$\tilde{b}_8 = \frac{1}{4xy}\left(b^{(a)} - b^{(c)} - b^{(b)} + b^{(d)}\right). \tag{5.98}$$

For the determination of the remaining coefficients, it is convenient to introduce two linear combinations

$$\begin{aligned} b_{36} &= \frac{1}{4x}\left(b^{(a)} - b^{(c)} + b^{(b)} - b^{(d)}\right), \\ b_{25} &= \frac{1}{2}\left(b^{(a)} - b^{(b)} - 2xyb_8\right).\end{aligned} \tag{5.99}$$

As the next step, we choose $x_1 = -0.5$. Note that this changes the value of $l_\perp^2$ according to Eq. (5.95). We then repeat the calculation described above. Our choices of momenta in the transverse plane $l_\perp$ are the same as in Eq. (5.97) but, to avoid confusion, we emphasize that x and y have to be calculated with the new $l_\perp^2$. We will refer to b computed with those new vectors as $\bar{b}^{(a)}$, $\bar{b}^{(b)}$, etc. We calculate $\bar{b}_{36,25}$ by substituting $b^{(\alpha)} \to \bar{b}^\alpha$ in Eq. (5.99). It is easy to see that simple linear combinations give the desired coefficients

$$\begin{aligned}\tilde{b}_3 &= \frac{1}{2}\left(b_{36} + \bar{b}_{36}\right), \quad \tilde{b}_6 = \frac{2}{s}\left(b_{36} - \bar{b}_{36}\right), \\ \tilde{b}_2 &= \frac{1}{2}\left(b_{25} + \bar{b}_{25}\right), \quad \tilde{b}_5 = \frac{2}{s}\left(b_{25} - \bar{b}_{25}\right).\end{aligned} \tag{5.100}$$

Other coefficients, required for the complete parametrization of the function $\tilde{b}(l)$ in Eq. (5.96), are obtained along similar lines; we do not discuss this further. However, we emphasize that the procedure that we just described is important for the computation of one-loop virtual amplitudes in a situation where both massless and massive particles are involved. In particular, it is heavily used in computations of NLO QCD corrections to top quark pair production discussed in Refs. [22, 23].

As a final remark, we note that there is another important difference between reducing the two-point function to scalar integrals for a light-like and a non-light-like vector. Consider only cut-constructible terms. Then, for $q_1^2 \neq 0$ integration over the transverse space can be immediately done, leading to

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\tilde{b}(l)}{d_0 d_1} = \tilde{b}_0 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{d_0 d_1}. \tag{5.101}$$

Hence, the only integral we need to know in $q_1^2 \neq 0$ case is the scalar two-point function. However, in case of a light-like vector $q_1^2 = 0$, *three* master integrals contribute to the cut-constructible part even after averaging over the directions of the vector l in the (two-dimensional) transverse space

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\tilde{b}(l)}{d_0 d_1} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\tilde{b}_0 + \tilde{b}_1(\bar{q}_1 \cdot l) + \tilde{b}_4(\bar{q}_1 \cdot l)^2}{d_0 d_1}. \tag{5.102}$$

Those integrals must be included in the basis of master integrals in the case when double cuts are considered with a light-like vector external vector. The calculation of those additional master integrals is straightforward. For completeness, we give the results below for the equal mass case $m_0 = m_1 = m$. We introduce $d_0 = l^2 - m^2$, $d_1 = (l + q_1)^2 - m^2$, $q_1^2 = 0$, $\bar{q}_1 q_1 = r$, $c_\Gamma = (4\pi)^{\epsilon-2}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2/\Gamma(1-2\epsilon)$ and find $(D = 4 - 2\epsilon)$

$$\frac{\mu^{2\epsilon}}{\mathrm{i}c_\Gamma} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l\bar{q}_1}{d_0 d_1} = -\frac{r}{2}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right), \tag{5.103}$$

$$\frac{\mu^{2\epsilon}}{\mathrm{i}c_\Gamma} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l\bar{q}_1)(l\bar{q}_1)}{d_0 d_1} = +\frac{r^2}{3}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right). \tag{5.104}$$

5.5 Comments on the rational part

The most general parametrizations of $\tilde{e}, \tilde{d}, \tilde{c}, \tilde{b}$ and $\tilde{a}$-functions contain two types of terms. First, there are terms that involve scalar products of the loop momenta with four-dimensional vectors from various transverse spaces. Second, there are terms that involve scalar products of the loop momentum with the $(D-4)$-dimensional components of the vectors spanning the transverse space. These latter terms require going beyond the four-dimensional loop momentum and give rise to the *rational part*.

6 Analytic techniques for one loop diagrams

6.1 Analytic Unitarity

The idea that unitarity can be used to calculate a loop integral is quite old. For example, Landau's book contains a dispersive calculation of the one-loop vertex function. However in the present context we do not want to perform the dispersive integral, rather we want to match the cut amplitudes, with a general set of scalar basis integrals, in such a way that we can identify the coefficients with which the basis integrals appear.

Britto et al. [24] have presented an efficient way to extract the box coefficients d_i by performing quadruple cuts. Performing a quadruple cut corresponding to a particular scalar box integral corresponds to replacing the each of the four propagators by

$$\frac{1}{l_i^2 + i\varepsilon} \to \delta^+(l_i^2) \tag{6.1}$$

This operation completely fixes all four components of the loop momentum. A general diagram will have numerator factors and in the case of higher point functions, also additional denominators, which we denote by $N(l)$

$$\begin{aligned} A &= \int \frac{d^4l}{(2\pi)^4} \frac{N(l)}{(l_0^2 + i\varepsilon)(l_1^2 + i\varepsilon)(l_2^2 + i\varepsilon)(l_3^2 + i\varepsilon)} \\ &\to \int \frac{d^4l}{(2\pi)^4}\, N(l)\ \delta^+(l_0^2)\delta^+(l_1^2)\delta^+(l_2^2)\delta^+(l_3^2) \\ &= \frac{1}{2} \sum_{\text{solutions}} N(l)\ . \end{aligned} \tag{6.2}$$

The sum is over the two solutions to the four simultaneous on-shell conditions.

If now apply the method to a complete amplitude, terms in the amplitude which do not contain all four of the cut denominators will not contribute to the discontinuity. This statement applies both to box or higher point integrals, which do not contains all four of the relevant propagators and to lower point functions, such as triangles which can contain at most three of the four propagators. Applying the same operation on the right-hand side of Eq. (**??**) neither the other box integrals, nor the lower point functions will contribute to the singularity. We can therefore read off the coefficient of the box integral in question. The overall result for the contribution of the given amplitude to the coefficient of a box integral is given by,

$$d_i = \frac{1}{2} \sum_{l_\pm} A_1^{\text{tree}}(l) A_2^{\text{tree}}(l) A_3^{\text{tree}}(l) A_4^{\text{tree}}(l) \tag{6.3}$$

where the A_i^{tree} are tree amplitudes at the four corners of the particular box in question.

Thus, in order to calculate all the box coefficients, we have to perform quadruple cuts corresponding to all box integrals which could be present.

6.2 Analytic methods

We will illustrate these methods by considering a specific case, for a simple 2 to 2 process. In particular we wish to calculate the one-loop amplitude for the process $\mathcal{A}_4(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+)$. The explicit decomposition of the $qgg\bar{q}$ one-loop amplitude is

$$\begin{aligned}\mathcal{A}_4(1_q, 2_g, 3_g, 4_{\bar{q}}) \quad &= g^4 \Big[N_c \, (T^{a_2} T^{a_3})_{i_1}^{\,\bar{\imath}_4} \; A_{4;1}(1_q, 2, 3, 4_{\bar{q}}) \; + \; N_c \, (T^{a_3} T^{a_2})_{i_1}^{\,\bar{\imath}_4} \; A_{4;1}(1_q, 3, 2, 4_{\bar{q}}) \\ &\qquad + \mathrm{Tr}(T^{a_2} T^{a_3}) \, \delta_{i_1}^{\,\bar{\imath}_4} \; A_{4;3}(1_q, 2, 3, 4_{\bar{q}}) \Big] \, , \end{aligned} \tag{6.4}$$

These colour stripped amplitudes can be further decomposed into primitive amplitudes,[25]

$$\begin{aligned} A_{4;1}(1_q, 2, 3, 4_{\bar{q}}) &= A_4^L(1_q, 2, 3, 4_{\bar{q}}) - \frac{1}{N_c^2} A_4^R(1_q, 2, 3, 4_{\bar{q}}) \\ + \quad &\frac{n_f}{N_c} A_4^{L,[1/2]}(1_q, 2, 3, 4_{\bar{q}}) + \frac{n_s}{N_c} A_4^{L,[0]}(1_q, 2, 3, 4_{\bar{q}}) \end{aligned} \tag{6.5}$$

The last two terms refer to terms with fermion or scalar loops. We will not consider either of these cases in this discussion. To further simplify the discussion we will only discuss the colour suppressed piece $A_4^R(1_q, 2, 3, 4_{\bar{q}})$. The relevant Feynman diagrams are shown in Fig. 6.1.

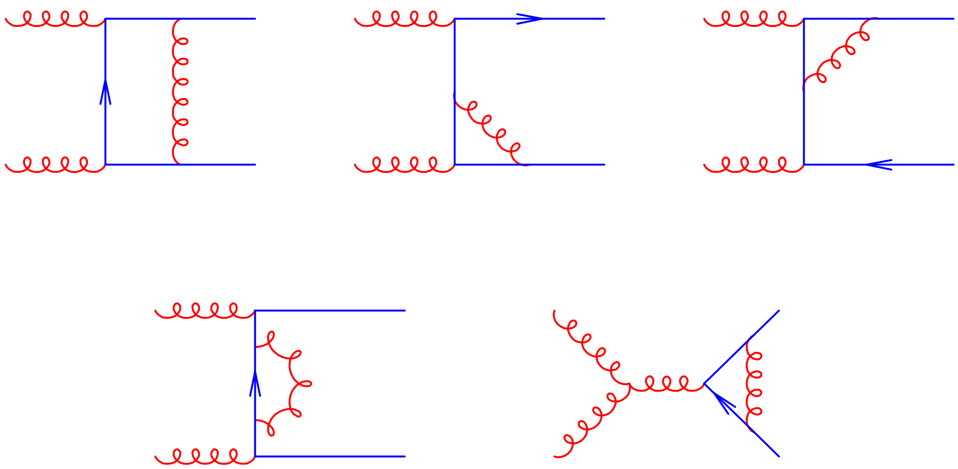


Figure 6.1: Feynman diagrams for the colour suppressed piece of the amplitude, $A_4^R(1_q, 2, 3, 4_{\bar{q}})$.

6.2.1 Example of the calculation of box coefficients

A simple example will illustrate the power of the method. Consider the diagram shown in Fig. 6.2. Momentum assignments are $l_2 = l_0 - p_2, l_{12} = l_0 - p_1 - p_2, l_3 = l_0 + p_3$ and all momenta are taken to be outgoing.

How do we parameterize the momenta which satisfy the four on-shell conditions? We choose to expand the vector l_0, (which is flanked on either side by two massless momenta

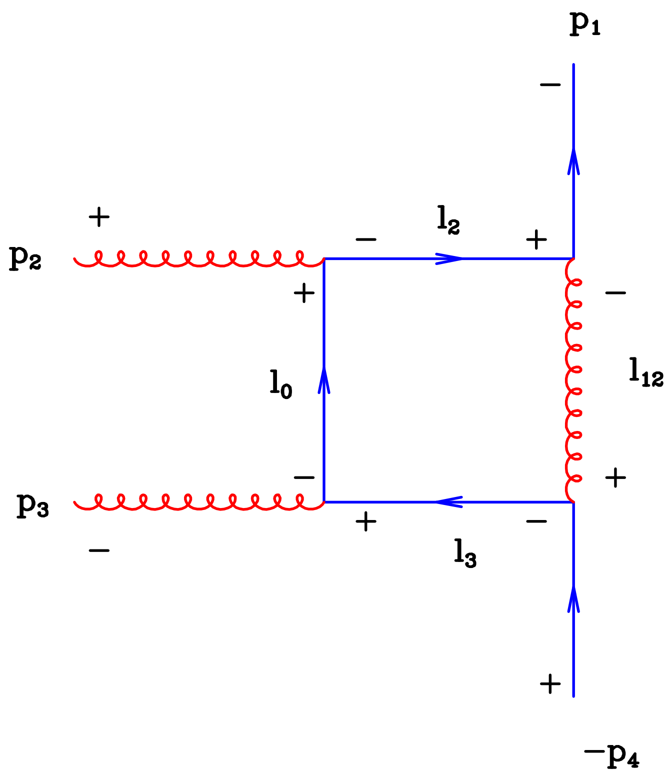


Figure 6.2: A box contribution to the amplitude $M(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+)$

p_2 and p_3), in terms of p_2 and p_3 as well as the complex momenta formed from spinors of p_2 and p_3.

$$l_0^\mu = \alpha p_2^\mu + \beta p_3^\mu + \gamma \epsilon_{23}^\mu + \delta \epsilon_{32}^\mu \tag{6.6}$$

where

$$\epsilon_{ij} = \frac{1}{2}\langle i - |\gamma^\mu| j - \rangle \tag{6.7}$$

For the purposes of this section we shall make the notation even more compact. Thus we write,

$$\begin{aligned} |i-\rangle &\rightarrow |i] \\ |i+\rangle &\rightarrow |i\rangle \\ \langle i - | &\rightarrow \langle i| \\ \langle i + | &\rightarrow [i| \end{aligned} \tag{6.8}$$

Now we use the mass shell conditions ($l_2^2 = l_3^2 = L_0^2 = l_{12}^2 = p_2^2 = p_3^2 = 0$). From the first two we obtain that $(l_0 - p_2)^2 \equiv -2l_0 \cdot p_2 = 0$ and $(l_0 + p_3)^2 \equiv 2l_0 \cdot p_3 = 0$, which tell us that $\alpha = 0$ and $\beta = 0$. The condition that $l_0^2 = 0$ tells us that the product $\gamma\delta = 0$, since

$$\begin{aligned} \epsilon_{ij} \cdot \epsilon_{ij} &= 0 \\ \epsilon_{ij} \cdot \epsilon_{ji} &= \frac{1}{2} s_{ij} \end{aligned} \tag{6.9}$$

We shall consider the choice $\delta = 0$; this we will justify a posteriori in the next section. Finally the condition that $(l_1 - p_1 - p_2)^2$ is on its mass shell gives us the condition,

$$- \gamma \, \langle p_2 | \not{p}_1 + \not{p}_2 | p_3] + 2 p_1 . p_2 = 0 \tag{6.10}$$

Hence we have that $\gamma = [1\,2]/[1\,3]$

6.3 Box coefficients for $qg g\bar{q}$

We consider first the case where the external gluons have the same helicity. This vanishes because we would have (++-) at both at the vertex where p_2 flows out and at the vertex where p_3 flows out. This implies that the two vertices are

$$\frac{[2\,l_0]^2}{[l_2\,l_0]} \times \frac{[3\,l_3]^2}{[l_0\,l_3]} \tag{6.11}$$

or consequently that both $\langle l_0\,2\rangle = 0$ and $\langle l_0\,3\rangle = 0$ which cannot simultaneously be satisfied for arbitrary external momenta p_2 and p_3. This helicity amplitude thus has a vanishing contribution to the coefficient of this box integral.

Next consider the case where the gluons have opposite helicities as shown in Fig. 6.2. Momentum assignments are $l_2 = l_0 - k_2, l_{12} = l_0 - k_1 - k_2, l_3 = l_0 + k_3$.

$$A_4^R(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+)|_{\text{box coeff}} = \frac{\langle l_{12}\,1\rangle^2}{\langle 1\,l_2\rangle} \times \frac{[2\,l_0]^2}{[l_2\,l_0]} \times \frac{\langle l_0\,3\rangle^2}{\langle l_0\,l_3\rangle} \times \frac{[4\,l_{12}]^2}{[4\,l_3]} \tag{6.12}$$

Since from Eq. (6.12) we have that $|l_0\rangle \sim |2\rangle, |l_0] \sim |3]$ in order that $\langle l_0\,2\rangle = 0$ and that $[l_0\,3] = 0$. The helicities of our diagrams pick the solution $\delta = 0$ so that we have.

$$l_0^\mu = \gamma \epsilon_{23}^\mu \tag{6.13}$$

or in other words that, up to an overall constant which cancels since l_0 always appears in the combination $\not{l}_0$,

$$|l_0\rangle = \gamma |2\rangle, \quad |l_0] = |3] \tag{6.14}$$

Collecting terms we may write Eq. (6.12) as,

$$\begin{aligned} M(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+) &= \frac{\langle 1|\not{l}_{12}|4]^2 \langle 3|\not{l}_0|2]^2}{\langle 1|\not{l}_2 \not{l}_0 \not{l}_3|4]} \\ &= \frac{\langle 1|\not{l}_0 - \not{2}|4]^2 \langle 3|\not{l}_0|2]^2}{\langle 1\,2\rangle [2|\not{l}_0|3\rangle [34]} \\ &= \frac{(\gamma \langle 1\,2\rangle [3\,4] - \langle 1\,2\rangle [2\,4])^2 \gamma \langle 3\,2\rangle [3\,2]}{\langle 1\,2\rangle [3\,4]} \end{aligned} \tag{6.15}$$

where we have made the substitution $\not{l}_0 = \gamma|2\rangle[3|$. Now from Eq. (6.10) $\gamma = [1\,2]/[1\,3]$, so we get

$$\begin{aligned}
A_4^R(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+)|_{\text{box coeff}} = & \frac{\langle 1\,2\rangle^2([1\,2][3\,4]-[1\,3][2\,4])^2[1\,2]\langle 3\,2\rangle[3\,2]}{\langle 1\,2\rangle[1\,3]^3[3\,4]} \\
= & \frac{\langle 1\,2\rangle[1\,4]^2[1\,2]\langle 3\,2\rangle[3\,2]^3}{[1\,3]^3[3\,4]} \\
= & \frac{\langle 1\,2\rangle\langle 1\,4\rangle[1\,4]^2[1\,2]\langle 3\,2\rangle[3\,2]^3}{[1\,3]^3[3\,4]\langle 1\,4\rangle} \\
= & \frac{\langle 1\,4\rangle[1\,4]^2[1\,2]\langle 3\,2\rangle[3\,2]^2}{[1\,3]^3} \\
= & \frac{s_{23}^2 s_{12}[1\,4][3\,2]}{\langle 1\,2\rangle[1\,3]^3} \\
= & \frac{s_{23}s_{12}[1\,4]^2[2\,3]^2}{[1\,3]^3[3\,4]}
\end{aligned} \tag{6.16}$$

In the above derivation we have been cavalier about signs and overall factors. Adjusting those, this result agrees with the result in Eq. (7.68).

7 Triangles, bubbles and rational terms

7.1 Triangles

7.1.1 Forde method for triangle coefficients

We will calculate the coefficient of the triangle integrals using the method of Forde [26]. We first review the case of three massless internal momenta, shown in Fig. 7.1 in order to introduce our notation which differs from that of Forde. Defining l_1 and l_2 as follows,

$$l_1^\mu = l_0^\mu - K_1^\mu \,, \qquad l_2^\mu = l_0^\mu + K_2^\mu \,, \tag{7.1}$$

the cut loop momenta $(l_i^2 = 0), i = 0, 1, 2$ may be written in the following general form

$$l_i^\mu = x_i K_1^{\flat,\mu} + y_i K_2^{\flat,\mu} + \frac{t}{2}\langle K_1^\flat|\gamma^\mu|K_2^\flat] + \frac{x_i y_i}{2t}\langle K_2^\flat|\gamma^\mu|K_1^\flat]. \tag{7.2}$$

All momenta can be expanded in terms of massless momenta,

$$\begin{aligned}
K_1 &= K_1^\flat + \frac{S_1}{\gamma}K_2^\flat \,, \\
K_2 &= K_2^\flat + \frac{S_2}{\gamma}K_1^\flat \,, \\
K_3 &= -(1+\frac{S_2}{\gamma})K_1^\flat - (1+\frac{S_1}{\gamma})K_2^\flat \,,
\end{aligned} \tag{7.3}$$

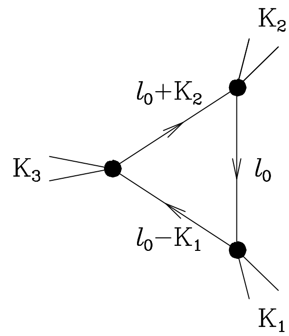


Figure 7.1: Triangle diagram showing the momentum parametrization. All momenta are outgoing, $K_1 + K_2 + K_3 = 0$

where $S_i = K_i^2$ and $\gamma = \langle K_1^\flat | K_2^\flat | K_1^\flat] = 2K_2^\flat . K_1^\flat$. The inverse relations are,

$$K_1^{\flat,\mu} = \frac{K_1^\mu - (S_1/\gamma) K_2^\mu}{1 - (S_1 S_2/\gamma^2)}, \qquad K_2^{\flat,\mu} = \frac{K_2^\mu - (S_2/\gamma) K_1^\mu}{1 - (S_1 S_2/\gamma^2)} \; . \tag{7.4}$$

γ can be expressed in terms of the external momenta,

$$\gamma_\pm = (K_1 \cdot K_2) \pm \sqrt{\Delta}, \qquad \Delta = (K_1 \cdot K_2)^2 - S_1 S_2. \tag{7.5}$$

The spinor solutions for the l_i can be expressed as a linear combination of the spinors for $K_1^\flat$ and $K_2^\flat$,

$$\langle l_i | = t \langle K_1^\flat | + y_i \langle K_2^\flat | \; , \qquad [l_i| = \frac{x_i}{t} [K_1^\flat | + [K_2^\flat | \; . \tag{7.6}$$

The on-shell conditions $l_i^2 = 0$ for $i = 0, 1, 2$ allow us to derive the coefficients, x_i and y_i,

$$\begin{aligned}
y_0 &= \frac{S_1\,(\gamma + S_2)}{(\gamma^2 - S_1 S_2)}, & x_0 &= -\frac{S_2\,(\gamma + S_1)}{(\gamma^2 - S_1 S_2)}, \\
y_1 = y_0 - \frac{S_1}{\gamma} &= \frac{S_1 S_2\,(\gamma + S_1)}{\gamma(\gamma^2 - S_1 S_2)}, & x_1 = x_0 - 1 &= -\frac{\gamma(\gamma + S_2)}{\gamma^2 - S_1 S_2}, \\
y_2 = y_0 + 1 &= \frac{\gamma(\gamma + S_1)}{\gamma^2 - S_1 S_2}, & x_2 = x_0 + \frac{S_2}{\gamma} &= -\frac{S_1 S_2\,(\gamma + S_2)}{\gamma(\gamma^2 - S_1 S_2)} \; .
\end{aligned} \tag{7.7}$$

The spinor products can be expressed as follows

$$\begin{aligned}
[l l_1] &= \frac{x_1 - x_0}{t}[K_2^\flat K_1^\flat] = -\frac{1}{t}[K_2^\flat K_1^\flat], \\
\langle l\, l_1\rangle &= t(y_1 - y_0)\langle K_1^\flat\, K_2^\flat\rangle = -\frac{tS_1}{\gamma}\langle K_1^\flat\, K_2^\flat\rangle, \\
[l l_2] &= \frac{x_2 - x_0}{t}[K_2^\flat K_1^\flat] = \frac{S_2}{\gamma t}[K_2^\flat K_1^\flat], \\
\langle l\, l_2\rangle &= t(y_2 - y_0)\langle K_1^\flat\, K_2^\flat\rangle = t\langle K_1^\flat\, K_2^\flat\rangle, \\
[l_1 l_2] &= \frac{x_2 - x_1}{t}[K_2^\flat K_1^\flat] = \frac{1}{t}\left(1 + \frac{S_2}{\gamma}\right)[K_2^\flat K_1^\flat], \\
\langle l_1\, l_2\rangle &= t(y_2 - y_1)\langle K_1^\flat\, K_2^\flat\rangle = t\left(1 + \frac{S_1}{\gamma}\right)\langle K_1^\flat\, K_2^\flat\rangle.
\end{aligned} \tag{7.8}$$

So far this is just algebra, that we have performed to impose the three mass shell constraints, $l_0^2 = (l - K_1)^2 = (l + K_2)^2 = 0$. From Eq. (7.2) we see that the parametrization of the momenta is of the form

$$l^\mu = a_0^\mu t + \frac{1}{t}a_1^\mu + a_2^\mu \tag{7.9}$$

Let us assume that there is another l dependent propagator, say of the form, $(l - P)^2$. From Eq.(7.9) we see that this fourth propagator, were it to go on shell, would lead to a term of the form,

$$(l - P)^2 = 0 \to -2t\, a_0 \cdot P - \frac{2}{t}\, a_1 \cdot P - 2\, a_2 \cdot P + P^2 = 0 \tag{7.10}$$

This quadratic in t will have two (complex) solutions that we can choose to partial fraction, leading the following expression for the product of amplitudes at the vertices of the triangle.

$$\int d^4 l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 = \int d^4 l \prod_{i=0}^{2} \delta(l_i^2)\left(J_t \sum_{i=0}^{m} f_i t^i + \sum_{poles\, j} \frac{\mathrm{Res}_{t=t_j} A_1 A_2 A_3}{t - t_j}\right) \tag{7.11}$$

The contribution of a triple-cut scalar box can be written down as follows

$$\int d^4 l \prod_{i=0}^{2} \delta(l_i^2)\frac{1}{(l-P)^2} \sim \frac{1}{t_+ - t_-}\left(\int d^4 l \prod_{i=0}^{2} \delta(l_i^2)\frac{1}{t - t_+} - \int d^4 l \prod_{i=0}^{2} \delta(l_i^2)\frac{1}{t - t_-}\right) \tag{7.12}$$

Therefore the terms with additional poles correspond exactly to the scalar box contributions. What about the remaining terms? What relation do they bear to the triangle coefficients. To provide the answer to this question we consider return to the parametrization of the momentum given in equation 7.2. We note first of all that

$$\begin{aligned}
&\langle K_1^{\flat,\pm}|\not{K}_1|K_2^{\flat,\pm}\rangle = 0 \\
&\langle K_2^{\flat,\pm}|\not{K}_1|K_1^{\flat,\pm}\rangle = 0 \\
&\langle K_1^{\flat,\pm}|\gamma^\mu|K_2^{\flat,\pm}\rangle\langle K_2^{\flat,\pm}|\gamma_\mu|K_1^{\flat,\pm}\rangle = 0
\end{aligned} \tag{7.13}$$

Let us now consider integrals of the form

$$\int d^4l \frac{\langle K_1^{\flat,-}|\not{l}|K_2^{\flat,-}\rangle^n}{l_0^2 l_1^2 l_2^2} = 0$$
$$\int d^4l \frac{\langle K_2^{\flat,-}|\not{l}|K_1^{\flat,-}\rangle^n}{l_0^2 l_1^2 l_2^2} = 0 \tag{7.14}$$

Both of these integrals must be equal to zero. The former implies that all negative powers of t integrate to zero, whereas the latter implies that all positive powers of t integrate to zero, as a specific consequence of the momentum parametrization, 7.2. Thus the result for the triangle coefficient is simply given by the coefficient of t^0.

7.1.2 Simple case

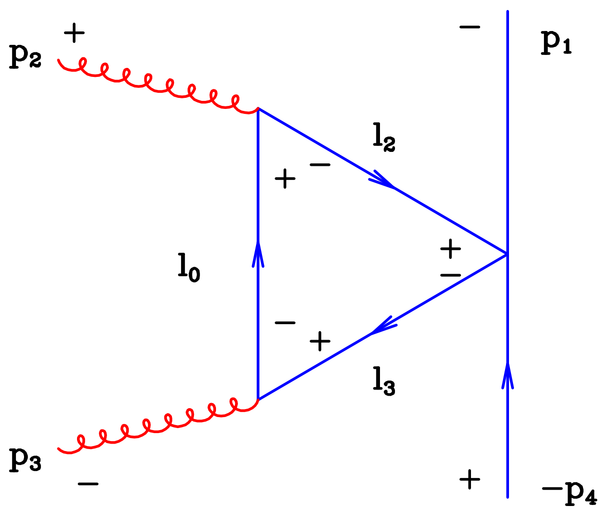


Figure 7.2: A triangle contribution

The neatest method to pick out the result for the coefficient of the triangle integral has been given by Forde [26]. We shall explain the method by referring to the particular case shown in Fig. 7.2. This case is particularly easy since two of the legs are light-like, but the method which we shall describe is able to handle more complicated cases. For details, see the paper of Forde [26]. As before we choose to expand the momentum in terms of

$$l_0^\mu = \alpha p_2^\mu + \beta p_3^\mu + \gamma \epsilon_{23}^\mu + \delta \epsilon_{32}^\mu \tag{7.15}$$

The conditions $(l_0 + p_3)^2 = (l_0 - p_2)^2 = 0$ fix $\alpha = \beta = 0$ as before. However now instead of two further conditions we have only one, namely $l_0^2 = 0$. This fixes the product of $\gamma\delta = 0$. For the same reason as before, in this case we are obliged to take the case $\delta = 0$, however γ is unconstrained.

$$l_0^\mu = \gamma \epsilon_{23}^\mu \tag{7.16}$$

Note however that

$$\int d^4l \, \frac{\langle 3|\not{l}_0|2]^n}{l_0^2 l_2^2 l_3^2} = 0 \to \int d\gamma \; J_\gamma \; \gamma^n \text{for} \quad n \geq 1$$

Because the momentum l_0 after integration can only give a term proportional to p_2, p_3 or $g^{\mu\nu}$ and $\langle 3|\not{3}|2] = \langle 3|\not{2}|2] = \langle 3|\gamma^\mu|2]\langle 3|\gamma_\mu|2] = 0$. Thus after removal of the box component which appears as a pole, the desired coefficient is obtained from the coefficient of γ^0. In general we can obtain the coefficient of the desired scalar integral by, taking the γ^0 component of the expansion of γ around infinity.

Results for triangles using the method of Forde. We shall need the amplitude for quark-quark scattering.

$$\begin{aligned} M(1_q^-, 2_{\bar{q}}^+, 3_Q^-, 4_{\bar{Q}}^+) &= \frac{\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4]}{\langle 2\,1\rangle[1\,2]} \\ &= 2\frac{\langle 1\,3\rangle[4\,2]}{\langle 2\,1\rangle[1\,2]} \end{aligned} \tag{7.17}$$

Hence we get

$$\begin{aligned} & \frac{[2\,l_0]^2}{[l_2\,l_0]} \times \frac{\langle l_0\,3\rangle^2}{\langle l_0\,l_3\rangle} \times \frac{\langle 1\,l_3\rangle[4\,l_2]}{\langle l_2\,1\rangle[1\,l_2]} \\ = & \frac{\langle 3|\not{l}_0|2]^2\langle 1\,l_3\rangle[4\,l_2][l_3\,2]\langle 3\,l_2\rangle}{\langle 3\,l_2\rangle[l_2\,l_0]\langle l_0\,l_3\rangle[l_3\,2]\langle l_2\,1\rangle[1\,l_2]} \\ = & \frac{\langle 3|\not{l}_0|2]^2\langle 1\,l_3\rangle[4\,l_2][l_3\,2]\langle 3\,l_2\rangle}{\langle 3|\not{l}_2\not{l}_0\not{l}_3|2]\langle l_2\,1\rangle[1\,l_2]} \\ = & \frac{\langle 3|\not{l}_0|2]^2\langle 1\,l_3\rangle[4\,l_2][l_3\,2]\langle 3\,l_2\rangle}{\langle 3\,2\rangle[2|\not{l}_0|3\rangle[3\,2]\langle l_2\,1\rangle[1\,l_2]} \\ = & \frac{\langle 3|\not{l}_0|2]\langle 1|\not{l}_3|2]\langle 3|\not{l}_2|4]}{\langle 3\,2\rangle[3\,2]\langle l_2\,1\rangle[1\,l_2]} \end{aligned} \tag{7.18}$$

where in the second line we have multiplied top and bottom by $[l_3\,2]\langle 3\,l_2\rangle$.

We now substitute the expression for the loop momentum which satisfies the three momentum constraints

$$l_0^\mu = \frac{\gamma}{2}\langle 2|\gamma^\mu|3] \tag{7.19}$$

The above expression now becomes

$$\frac{\gamma(\gamma\langle 1\,2\rangle + \langle 1\,3\rangle)(\gamma[3\,4] - [2\,4])[3\,2]\langle 3\,2\rangle}{\langle 1\,2\rangle(\gamma[3\,1] - [2\,1])} \tag{7.20}$$

We can partial fraction this result, to remove the box contribution, which obviously corresponds to the pole; evaluating the remainder at $\gamma = 0$ and using the Schouten Identity Eq. 1.28 we obtain

$$\frac{-s_{23}^2[1\,4][3\,2]}{\langle 1\,2\rangle[3\,1]^3} = \frac{s_{23}[1\,4]^2[2\,3]^2}{[3\,1]^3[3\,4]} \tag{7.21}$$

Up to an overall constant this is the result given for $C_0(p_2, p_3, 0, 0, 0)$ in the full answer, shown in a later section.

7.2 The bubble coefficients

7.2.1 General methods

Techniques exist to obtain analytic results for the bubble [26, 27] and rational terms [28]. Here we shall discuss a method of picking out bubble coefficients due to Mastrolia. Let us consider the integral with two cut propagators in the centre of mass frame of P

$$\int d^4l\delta^+(l^2)\delta^+((l-P)^2)f(l) = \frac{1}{8}\int d\Omega f(l) \tag{7.22}$$

where $d\Omega$ is the integration over the unit sphere. So in the centre of mass frame of P, after imposing the delta function constraints, we only have to integrate over θ and ϕ. Now since we are interested in isolating the poles in the function $f(l)$ it is convenient to get rid of the transcendental functions by performing the usual half-angle substitution.

$$\tau = \tan\frac{\theta}{2}, \quad \cos\theta = \frac{1-\tau^2}{1+\tau^2}. \quad \sin\theta = \frac{2\tau}{1+\tau^2}, \rho = \exp(i\phi) \tag{7.23}$$

In terms of these variables we can write

$$\frac{1}{4\pi}\int d\Omega = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \to \int_0^\infty d\tau \frac{2\tau}{(1+\tau^2)^2}\frac{1}{2\pi i}\oint_{|\rho|=1}\frac{d\rho}{\rho} \tag{7.24}$$

Defining $z = \tau\rho, \bar{z} = \tau/\rho$ we obtain

$$\int \frac{d\Omega}{4\pi} = \frac{1}{2\pi i}\int\int_D dz d\bar{z}\frac{1}{(1+z\bar{z})^2} \tag{7.25}$$

where the integration is over the whole complex plane, D. It is convenient also to examine the decomposition of the momentum l. We decompose the momentum $P = p+q$ in terms of two massless momenta p and q. In the centre of mass frame of P we have that,

$$\begin{aligned} p^\mu &= E(1,0,0,1) \\ q^\mu &= E(1,0,0,-1) \\ \varepsilon_{pq} = \frac{1}{2}\langle p|\gamma^\mu|q] &= E(0,1,i,0) \\ \varepsilon_{qp} = \frac{1}{2}\langle q|\gamma^\mu|p] &= E(0,1,-i,0) \end{aligned} \tag{7.26}$$

Parameterizing l as follows

$$l = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{7.27}$$

we find that

$$l^\mu = \frac{1}{1+z\bar{z}}(p^\mu + z\bar{z}q^\mu + \varepsilon^\mu_{qp}z + \varepsilon^\mu_{pq}\bar{z}) \tag{7.28}$$

We need the develop some familiarity dealing with integrals of the form in Eq. (7.25). One way to perform the integration is to transform back to real variables. Thus making the transformations $z \to x+iy, \bar{z} \to x-iy$,

$$\frac{d\Omega}{4\pi} = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(1+z\bar{z})^2} \tag{7.29}$$

replacing $z\bar{z}$ with $x^2 + y^2$ we can immediately perform the integration. This is not the direction we pursue. Rather we want to perform the integration directly in the complex plane using an extension of Cauchy's theorem, known as the *Generalised Cauchy Formula* or *Cauchy-Pompeiu Formula*.

Let f be an arbitrary function in a finite closed domain D bounded by a piecewise-smooth curve L, then the Cauchy-Pompeiu formula states that[29]

$$\begin{aligned} &\frac{1}{2\pi i}\int_L dz\,\frac{f(z)}{z-z_0} - \frac{1}{\pi}\int\int_D f_{\bar{z}}\,\frac{dxdy}{z-z_0} = f(z_0), \quad z_0 \in D \\ &f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big), \quad z = x + iy \end{aligned} \tag{7.30}$$

Let us first consider the case when $f(z)$ is analytic. Then $f_{\bar{z}} = 0$, because an analytic function cannot depend on both z and $\bar{z}$. For this case we recover the standard Cauchy integral formula,

$$\frac{1}{2\pi i}\int_L \frac{f(z)dz}{z-z_0} = f(z_0), \quad z_0 \in D \tag{7.31}$$

Second, let us consider the case where f vanishes on the boundary of D, denoted by L, so that we can drop the first term on the LHS of Eq. (7.30),

$$\frac{1}{\pi}\int\int_D f_{\bar{z}}\,\frac{dxdy}{z_0-z} = f(z_0), \quad z_0 \in D\,. \tag{7.32}$$

So in general, to perform the integration of a function of the form

$$\int\int_D dz d\bar{z}\; g(z,\bar{z}) \tag{7.33}$$

The first step[3] is to identify the primitive of the function $g(z,\bar{z})$ with respect to $\bar{z}$, keeping z constant,

$$G(z,\bar{z}) = \int d\bar{z}\; g(z,\bar{z}) \tag{7.34}$$

Thus at this point we have written the integrand in the form

$$\frac{1}{\pi}\int\int_D dz d\bar{z}\; G_{\bar{z}}, \quad G_{\bar{z}} = \frac{\partial G}{\partial \bar{z}}. \tag{7.35}$$

Before proceeeding to perform the integration, let us examine the structure of G. Since G is the primitive of a rational function, g, in general it can only contain two types of terms,

$$G(z,\bar{z}) = G^{\mathrm{rat}}(z,\bar{z}) + G^{\mathrm{log}}(z,\bar{z}) \tag{7.36}$$

Now the double cut of a two-point scalar function is rational, whereas the double cut of higher-point scalar integrals will contain logarithms. Since our aim is to isolate the contribution from the two-point function, we can drop the logarithmic terms and retain only the rational piece. We will return to this point in the next subsection when we consider a simple example.

[3]Note that we could as well do it for z.

In the case at hand, Eq. (7.29), (i.e. the scalar two-point integral) we obviously have no contributions from higher-point integrals. We take

$$g(z,\bar{z}) = \frac{1}{(1+z\bar{z})^2} \tag{7.37}$$

and as expected the primitive of g contains no logarithms.

$$G(z,\bar{z}) = -\frac{1}{z}\frac{1}{(1+z\bar{z})} \tag{7.38}$$

Now performing the identifications

$$f_z = \frac{1}{1+z\bar{z}}, \quad z_0 = 0 \tag{7.39}$$

we can use Eq. (7.32) to perform the integral in Eq. (7.25) to recover the standard result for the angular integral. In general, in addition to the pole at $z = 0$ we will have other poles at $z \neq 0$ indicating the contributions from higher point tensor integrals to the two-point function.

7.2.2 Simple example

A four momentum satisfying the two on-mass shell conditions in Eq. (7.22) can be written as

$$l^\mu = \frac{1}{1+z\bar{z}}(p_2^\mu + z\bar{z}p_3^\mu + \varepsilon_{32}^\mu z + \varepsilon_{23}^\mu \bar{z}) \tag{7.40}$$

where

$$\varepsilon_{ij}^\mu = \frac{1}{2}\langle i|\gamma^\mu|j] \tag{7.41}$$

It is convenient to remove an overall scale, so we define a rescaled momentum. $\langle l| = \sqrt{t}\langle\lambda|,\ |l] = \sqrt{t}|\lambda]$ where

$$t = \frac{1}{(1+z\bar{z})} = \frac{P^2}{\langle\lambda|P|\lambda]} \tag{7.42}$$

so that the rescaled momentum λ is given by,

$$\lambda^\mu = p_2^\mu + z\bar{z}p_3^\mu + \varepsilon_{32}^\mu z + \varepsilon_{23}^\mu \bar{z} \tag{7.43}$$

As a warm up let us consider how various integrands appear when expressed in terms of z and $\bar{z}$.

$$\begin{aligned}
\frac{1}{l^2(l-p_2-p_3)^2} &\to \frac{1}{(1+z\bar{z})^2} \\
\frac{1}{l^2(l-p_2)^2(l-p_2-p_3)^2} &\to -\frac{1}{2p_2\cdot p_3}\frac{1}{(1+z\bar{z})z\bar{z}} \\
\frac{l^\mu}{l^2(l-p_2)^2(l-p_2-p_3)^2} &\to -\frac{1}{2p_2\cdot p_3}\frac{p_2^\mu + z\bar{z}p_3^\mu + z\varepsilon_{32} + \bar{z}\varepsilon_{23}}{(1+z\bar{z})^2 z\bar{z}}
\end{aligned} \tag{7.44}$$

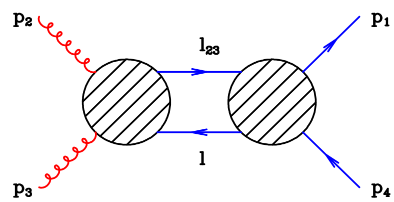


Figure 7.3: The double cut of the amplitude $M^{(1)}(1_q^-1, 2_q^+, 3_Q^-, 4_Q^+)$

Obtaining the primitives with respect to $\bar{z}$ we get

$$\begin{aligned}
\frac{1}{l^2(l-p_2-p_3)^2} &\to -\frac{1}{z(1+z\bar{z})} \\
\frac{1}{l^2(l-p_2)^2(l-p_2-p_3)^2} &\to -\frac{1}{2p_2\cdot p_3}\frac{\ln(\bar{z})-\ln(z\bar{z}+1)}{z} \\
\frac{l^\mu}{l^2(l-p_2)^2(l-p_2-p_3)^2} &\to -\frac{1}{2p_2\cdot p_3}\frac{(p_3^\mu-p_2^\mu)-z\varepsilon_{32}^\mu+\frac{1}{z}\varepsilon_{23}^\mu}{(1+z\bar{z})z} \\
&+ \text{ logarithmic terms}
\end{aligned} \tag{7.45}$$

Discarding the logarithmic terms we find the expected contributions to the bubble coefficients coming from the scalar bubble and the rank-one triangle.

7.2.3 Application to $A_4^R(1_q^-, 2_q^+, 3_Q^-, 4_Q^+)$

Now we turn to the concrete physical example, shown in Fig. 7.3. First we should write down the amplitude for four-quark scattering, cf Eq. (7.17).

$$M^{(0)}(1_q^-, 2_q^+, 3_Q^-, 4_Q^+) = \frac{1}{2}\langle p_1|\gamma^\mu|p_2]\langle p_3|\gamma_\mu|p_4]\frac{1}{(p_1+p_2)^2} \tag{7.46}$$

$$= \frac{\langle p_1\, p_3\rangle[p_4\, p_2]}{\langle p_1\, p_2\rangle[p_2\, p_1]} \tag{7.47}$$

$$= \frac{\langle p_1\, p_3\rangle[p_4\, p_2]\langle p_4\, p_3\rangle}{\langle p_1\, p_2\rangle[p_2\, p_1]\langle p_4\, p_3\rangle} \tag{7.48}$$

$$= \frac{\langle p_1\, p_3\rangle[p_2\, p_1]\langle p_1\, p_3\rangle}{\langle p_1\, p_2\rangle[p_2\, p_1]\langle p_4\, p_3\rangle} \tag{7.49}$$

$$= \frac{\langle p_1\, p_3\rangle^2}{\langle p_1\, p_2\rangle\langle p_4\, p_3\rangle} \equiv \frac{[2\,4]^2}{[1\,2][3\,4]} \tag{7.50}$$

In addition we shall need the amplitude for opposite helicity gluon quark scattering.

$$M^{(0)}(1_q^+, 2_g^+, 3_g^-, 4_{\bar{q}}^-) = \frac{\langle 3\,1\rangle^3\langle 3\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \tag{7.51}$$

Now we form the combination as indicated in Fig. 7.3

$$M_L(-l, p_2, p_3, l_{23}) \times M_R(p_1, l, -l_{23}, p_4) = \frac{\langle 3\,l\rangle^3}{\langle l\,2\rangle\langle 2\,3\rangle\langle l_{23}\,l\rangle} \times \frac{[4\,l]^2}{[1\,l][4\,l_{23}]} \tag{7.52}$$

We eliminate l_{23} using the momentum conservation relation,

$$\langle l\, l_{23}\rangle[l_{23}\, 4] = -\langle l|P|4] = \langle l\, 1\rangle[1\, 4] \tag{7.53}$$

So we obtain

$$M_L \times M_R \;=\; \frac{1}{\langle 2\, 3\rangle[1\, 4]} \times \frac{\langle 3\, l\rangle^3[4\, l]^2}{\langle 1\, l\rangle[1\, l]\langle 2\, l\rangle} \tag{7.54}$$

Putting in the integration measure and rescaling using Eq.(7.42)

$$\begin{aligned} b = \int d^4 l\delta^+(l^2)\delta^+((l-P)^2)\; M_L \times M_R \\ = \int \frac{dz d\bar{z}}{(1+z\bar{z})}\frac{\langle 1\, 4\rangle}{\langle 2\, 3\rangle} \times \frac{s_{23}\langle 3\, \lambda\rangle^3[4\, \lambda]^2}{(1+z\bar{z})[1\, \lambda]\langle 1\, \lambda\rangle\langle 2\, \lambda\rangle} \end{aligned} \tag{7.55}$$

It now is useful to apply the Mastrolia expansion with vectors p_2, p_3

$$\lambda^\mu = p_2^\mu + z\bar{z}p_3^\mu + \frac{z}{2}\langle 3|\gamma^\mu|2] + \frac{\bar{z}}{2}\langle 2|\gamma^\mu|3] \tag{7.56}$$

Thus we have that

$$\begin{aligned} |\lambda] &= |2] + \bar{z}|3] \\ \langle\lambda| &= \langle 2| + z\langle 3| \end{aligned} \tag{7.57}$$

To evaluate the result we shall first find the primitive with respect to z

$$b = \frac{s_{23}^3}{\langle 2\, 3\rangle[1\, 4]} \times \oint_C d\bar{z} \int dz \frac{\langle 3\, \lambda\rangle^3[4\, \lambda]^2}{\langle\lambda|P|\lambda]^3[1\, \lambda]\langle 1\, \lambda\rangle\langle 2\, \lambda\rangle} \tag{7.58}$$

The terms containing $\langle\lambda|$ are of the form

$$A = \frac{\langle 3\, \lambda\rangle^3}{\langle\lambda|P|\lambda]^3\langle 1\, \lambda\rangle\langle 2\, \lambda\rangle} \tag{7.59}$$

Using Eq. (7.57) we make the following simplifications of the terms in A

$$\begin{aligned} \langle 3\, \lambda\rangle &= -\langle 2\, 3\rangle \\ \langle 1\, \lambda\rangle &= \langle 1\, 2\rangle + z\langle 1\, 3\rangle \\ \langle 2\, \lambda\rangle &= z\langle 2\, 3\rangle \\ \langle\lambda|P|\lambda] &= \langle 2\, 3\rangle([3\, \lambda] - z[2\, \lambda]) \end{aligned} \tag{7.60}$$

Thus Eq. 7.59 becomes

$$A = -\frac{1}{z\langle 2\, 3\rangle}\frac{1}{(z[2\, \lambda] - [3\, \lambda])^3(z\langle 1\, 3\rangle + \langle 1\, 2\rangle)} \tag{7.61}$$

By partial fractioning, obtaining the primitve with respect to z and dropping the logarithmic terms we get

$$+\frac{[2\, \lambda](2\langle 1|2|\lambda] + \langle 1|3|\lambda])}{\langle 2\, 3\rangle[3\, \lambda]^2\langle 1|2+3|\lambda]^2([2\, \lambda]z - [3\, \lambda])} - \frac{[2\, \lambda]}{2\langle 2\, 3\rangle[3\, \lambda]\langle 1|2+3|\lambda]([2\, \lambda]z - [3\, \lambda])^2} \tag{7.62}$$

We now multiply by the missing factors

$$\frac{s_{23}^3}{\langle 2\,3\rangle[1\,4]} \times \frac{[4\,\lambda]^2}{[1\,\lambda]} \tag{7.63}$$

and make substitutions from Eq. (7.57)

$$\begin{aligned}
[1\,\lambda] &= [1\,2] + \bar{z}[1\,3] \\
[2\,\lambda] &= \bar{z}[2\,3] \\
[3\,\lambda] &= -[2\,3] \\
[4\,\lambda] &= -[2\,4] - \bar{z}[3\,4] = -[2\,4] + \bar{z}\frac{\langle 1\,2\rangle[2\,4]}{\langle 1\,3\rangle}
\end{aligned} \tag{7.64}$$

Taking the residue at $\bar{z} = -[1\,2]/[1\,3]$ which is the only pole that gives a contribution and setting $z = \langle 1\,2\rangle/\langle 1\,3\rangle$ we get

$$\frac{[1\,2](5\langle 1\,3\rangle[1\,3] + 2\langle 1\,2\rangle[1\,2])[2\,4]^2}{2\langle 1\,3\rangle\langle 2\,3\rangle[1\,3]^2(\langle 1\,3\rangle[1\,3] + \langle 1\,2\rangle[1\,2])[2\,3]^3} \tag{7.65}$$

Including all factors we obtain for the $B_0(p_{23}, 0, 0)$ bubble contribution to the amplitude,

$$M^{(1)}(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+) = -\left[\frac{6[1\,2][2\,4]^2}{2[1\,3][1\,4][2\,3]} + 4\frac{[1\,2][2\,4]}{[1\,3]^2}\right] B_0(p_{23}, 0, 0) \tag{7.66}$$

7.3 Gluonic result

Up to a sign the result for the lowest order cross section is, c.f. Eq.(7.51)

$$A_4^{\rm tree}(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+) \;=\; 4\frac{[2\,4]^3}{[2\,3][1\,4][3\,4]} \tag{7.67}$$

The full answer for the cut constructible part of the $1/N$ piece of this amplitude is

$$\begin{aligned}
A_4^R(1_q^-, 2_g^+, 3_g^-, 4_{\bar{q}}^+) \;=\; & -2\frac{[2\,3]^2[1\,4]^2}{[1\,3]^3[3\,4]}\Big[s_{12}s_{23}D_0(p_1, p_2, p_3, 0, 0, 0, 0) - s_{12}C_0(p_1, p_2, 0, 0, 0) \\
& - s_{12}C_0(p_{12}, p_3, 0, 0, 0) - s_{23}C_0(p_2, p_3, 0, 0, 0) - s_{23}C_0(p_1, p_{23}, 0, 0, 0)\Big] \\
& - 4\frac{[2\,4]^3}{[2\,3][1\,4][3\,4]}\Big[s_{23}C_0(p_1, p_{23}, 0, 0, 0) + \frac{3}{2}B_0(p_{23}, 0, 0)\Big] \\
& + \Big[4\frac{[1\,2][2\,4]}{[1\,3]^2} - 6\frac{[2\,4]^2}{[1\,3][3\,4]}\Big]\Big[B_0(p_{12}, 0, 0) - B_0(p_{23}, 0, 0)\Big] \\
& - 2\frac{[1\,2][2\,4]^2}{[1\,3][2\,3][1\,4]}
\end{aligned} \tag{7.68}$$

7.4 Singular behaviour at one-loop order

The results the infrared-singular behaviour of QCD amplitudes at one-loop order with massless particles have been given in ref. [30]. The results are given in color space notation, which has the advantage that it can deal with both quarks and gluons in a seamless way. An explanation of the colour operators acting on simple state is given in Appendix A of ref.[31]. The one-loop subamplitude $\mathcal{M}_m^{(1)}(\mu^2;\{p\})$ has double and single poles in $1/\epsilon$. The coefficients of these poles are given by the following formula

$$|\mathcal{M}_m^{(1)}(\mu^2;\{p\})\rangle_{\rm R.S.} = \boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\ |\mathcal{M}_m^{(0)}(\mu^2;\{p\})\rangle_{\rm R.S.} + O(\epsilon) \tag{7.69}$$

We see that one-loop singularities are factorized in color space with respect to the tree-level amplitude $\mathcal{M}_m^{(0)}$. The singular dependence is given by the factor $\boldsymbol{I}^{(1)}$ that acts as a colour-charge operator onto the colour vector $|\mathcal{M}_m^{(0)}\rangle$.

$\boldsymbol{I}^{(1)}$ has the following explicit expression in terms of colour charges of the m quarks and gluons that participate in the amplitude.

$$\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) = g^2 c_\Gamma \sum_i \frac{1}{\boldsymbol{T}_i^2}\,\mathcal{V}_i^{\rm sing}(\epsilon)\ \sum_{j\neq i} \boldsymbol{T}_i\cdot\boldsymbol{T}_j\ \left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i\cdot p_j}\right)^\epsilon \quad , \tag{7.70}$$

where $e^{-i\lambda_{ij}\pi}$ is the unitarity phase ($\lambda_{ij}=+1$ if i and j are both incoming or outgoing partons and $\lambda_{ij}=0$ otherwise) and the singular (for $\epsilon\to 0$) function $\mathcal{V}_i^{\rm sing}(\epsilon)$ depends only on the parton flavour and is given by

$$\mathcal{V}_i^{\rm sing}(\epsilon) = \boldsymbol{T}_i^2\frac{1}{\epsilon^2} + \gamma_i\ \frac{1}{\epsilon} \quad . \tag{7.71}$$

The flavour coefficients $\boldsymbol{T}_i^2$ and γ_i are

$$\begin{aligned} &\boldsymbol{T}_q^2 = \boldsymbol{T}_{\bar q}^2 = C_F \ , && \boldsymbol{T}_g^2 = C_A \ , \\ &\gamma_q = \gamma_{\bar q} = \frac{3}{2}C_F \ , && \gamma_g = \frac{b0}{2} = \frac{11}{6}C_A - \frac{2}{3}T_R N_f \ . \end{aligned} \tag{7.72}$$

Note that in Eq. (7.69) the double poles $1/\epsilon^2$ are factorized completely. If we expand Eq. (7.70) in powers of ϵ and then use the colour conservation relation, $\sum_{j\neq i}\boldsymbol{T}_j = -\boldsymbol{T}_i$. One obtains the result

$$\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) = g^2\sum_i \frac{1}{\epsilon^2}\ \sum_{j\neq i}\boldsymbol{T}_i\cdot\boldsymbol{T}_j + \mathcal{O}(1/\epsilon) = -\frac{1}{2\epsilon^2}\sum_i \boldsymbol{T}_i^2 + \mathcal{O}(1/\epsilon) \ , \tag{7.73}$$

that explicitly shows the absence of colour correlations at $\mathcal{O}(1/\epsilon^2)$. Note that the single poles $1/\epsilon$ will have colour correlations.

An important check on any calculation is that it reproduces the correct result for the $1/\epsilon^2, 1/\epsilon$ poles. The form of the singular behaviour has been given by Catani and collaborators in ref. [30] for the case of massless quarks and for massive quarks in ref.[32].

7.5 Assembling it all: Inserting the integrals

The result for the box integral with all external lines light-like is,

$$
\begin{aligned}
& I_4^{\{D=4-2\epsilon\}}(0,0,0,0;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{12}s_{23}} \\
\times \quad & \left[\frac{2}{\epsilon^2}\Big((-s_{12}-i\varepsilon)^{-\epsilon}+(-s_{23}-i\varepsilon)^{-\epsilon}\Big) - \ln^2\left(\frac{-s_{12}-i\varepsilon}{-s_{23}-i\varepsilon}\right) - \pi^2\right] + \mathcal{O}(\epsilon)\,. \qquad (7.74)
\end{aligned}
$$

Triangle with two massless external lines

$$
\begin{aligned}
I_3^{\{D=4-2\epsilon\}}(0,0,p^2;0,0,0) &= \frac{\mu^{2\epsilon}}{\epsilon^2}\left(\frac{(-p^2-i\varepsilon)^{-\epsilon}}{p^2}\right) \\
&= \frac{1}{p^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\left(\frac{\mu^2}{-p^2-i\varepsilon}\right)+\frac{1}{2}\ln^2\left(\frac{\mu^2}{-p^2-i\varepsilon}\right)\right)+\mathcal{O}(\epsilon) \qquad (7.75)
\end{aligned}
$$

Indeed since the triangle integral above, only provides a double pole, its coefficient can often be guessed without calculation, in order to reproduce the known IR behaviour.

$$
\begin{aligned}
I_2^{\{D=4-2\epsilon\}}(p^2;0,0) &= \left(\frac{\mu^2}{-p^2-i\varepsilon}\right)^{\epsilon}\left(\frac{1}{\epsilon}+2\right) \\
&= \frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{-p^2-i\varepsilon}\right)+2+\mathcal{O}(\epsilon). \qquad (7.76)
\end{aligned}
$$

The final result for the color supressed in the one loop $qgg\bar{q}$ amplitude, can be obtained by substituting these integrals in Eq. (7.68),

$$
\begin{aligned}
& A_4^R(1_q^-,2_g^+,3_g^-,4_{\bar{q}}^+) = g^2(\mu^2)^{\epsilon}c_\Gamma\frac{4[2\,4]^3}{[1\,4][2\,3][3\,4]}\Bigg[-\frac{1}{\epsilon^2}\Big(-s_{23}\Big)^{-\epsilon}-\frac{3}{2\epsilon}(-s_{23})^{-\epsilon}-\frac{7}{2} \\
- \quad & \frac{1}{2}\frac{s_{23}}{s_{13}}\left[\left(1-\frac{s_{23}}{s_{13}}\ln\left(\frac{s_{12}}{s_{23}}\right)\right)^2+\ln\left(\frac{s_{12}}{s_{23}}\right)+\pi^2\frac{s_{23}^2}{s_{13}^2}\right]\Bigg]\,. \qquad (7.77)
\end{aligned}
$$

and c_Γ is

$$
c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \qquad (7.78)
$$

After taking acccount of the different regularization scheme (four-dimensional helicity vs 't Hooft-Veltman) that we are using, this result is in agreement with ref. [33].

7.6 Rational terms: Axial anomaly

The material in this section is taken from **??**. One of the manifestations of the Adler-Bell-Jackiw axial anomaly in QED [34, 35] is the peculiar property of the matrix element of the divergence of the axial current $J_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi$, where ψ is the "electron" field, taken between the vacuum and the two-photon states. For massless electrons, such a matrix element reads

$$\mathcal{M}_{ABJ} = \langle \gamma(k_1,\lambda_1)\gamma(k_2,\lambda_2) \mid \partial^\mu J^5_\mu(0) \mid 0\rangle \; = \; \frac{e^2}{2\pi^2}\varepsilon^{\mu\nu\lambda\rho} e^*_{1\mu} k_{1\nu} e^*_{2\lambda} k_{2\rho}, \tag{7.79}$$

where $k_{1,2}$ and $e_{1,2}$ are momenta and polarization vectors of the outgoing photons with helicities $\lambda_{1,2}$. The matrix element $\mathcal{M}_{ABJ}$ is purely rational. Below we derive $\mathcal{M}_{ABJ}$ using the algorithm of D-dimensional unitarity.

The amplitude M_{ABJ} is given by the sum of two triangle Feynman diagrams with the electron loop. The matrix element is written as

$$\begin{aligned} \mathcal{M}_{ABJ} = & \frac{ie^2}{(4\pi)^{(D/2)}} \int \frac{d^D l}{i(\pi)^{(D/2)}} \\ & \times \mathrm{Tr}\left\{ \hat{k}_{12}\Gamma_{\gamma_5}\left[\frac{\hat{l}\hat{e}^*_1(\hat{l}+\hat{k}_1)\hat{e}^*_2(\hat{l}+\hat{k}_{12})}{l^2(l+k_1)^2(l+k_{12})^2}\right] + (1\leftrightarrow 2)\right\}, \end{aligned} \tag{7.80}$$

where $k_{12} = k_1 + k_2$. The external momentum and polarization vectors are four-dimensional, whereas the loop momentum and the Dirac matrices Γ^μ and the matrix Γ_{γ_5} are continued to D-dimensions, following the discussion in Section **??**. We note that Γ_{γ_5} in Eq.(7.80) denotes the D-dimensional continuation of the matrix γ_5. We perform such a continuation following t'Hooft and Veltman [5]. It is defined by the set of commutation relations

$$\begin{aligned} &\{\Gamma^\mu, \Gamma_{\gamma_5}\} = 0, \quad \text{for } \mu = 0,1,2,3 \\ &[\Gamma^\mu, \Gamma_{\gamma_5}] = 0. \;\; \text{for } \mu = 4,\dots D-1 \end{aligned} \tag{7.81}$$

Equation (7.80) defines the integrand function of the loop momentum integral but it does not define it uniquely. This is not a problem since the integral is regularized dimensionally and shifts of the loop momenta are allowed. We will exploit such shifts to simplify the computation. To this end, we split the integrand in Eq. (7.80) using the identity

$$\hat{k}_{12}\Gamma_{\gamma_5} \; = \; (\hat{l}+\hat{k}_{12})\Gamma_{\gamma_5} + \Gamma_{\gamma_5}\hat{l} - 2\Gamma_{\gamma_5}\hat{l}_\epsilon, \tag{7.82}$$

where $l_\epsilon = (l\cdot n_\epsilon)n_\epsilon$ is the $(D-4)$-dimensional part of the loop momentum. After the split, the trace in Eq.(7.80) gets additional terms

$$\begin{aligned} &\mathrm{Tr}\left\{\hat{k}_{12}\Gamma_{\gamma_5}\left[\frac{\hat{l}\hat{e}^*_1(\hat{l}+\hat{k}_1)\hat{e}^*_2(\hat{l}+\hat{k}_{12})}{l^2(l+k_1)^2(l+k_{12})^2}\right] + (1\leftrightarrow 2)\right\} = \mathrm{Tr}_1 + \mathrm{Tr}_2, \\ &\mathrm{Tr}_1 = -2\mathrm{Tr}\left\{\Gamma_{\gamma_5}\hat{l}_\epsilon \frac{\hat{l}\hat{e}^*_1(\hat{l}+\hat{k}_1)\hat{e}^*_2(\hat{l}+\hat{k}_{12})}{l^2(l+k_1)^2(l+k_{12})^2} + (1\leftrightarrow 2)\right\}, \\ &\mathrm{Tr}_2 = \mathrm{Tr}\left\{\Gamma_{\gamma_5}\left[\frac{\hat{l}\hat{e}^*_1(\hat{l}+\hat{k}_1)\hat{e}^*_2}{l^2(l+k_1)^2} + \frac{\hat{e}^*_1(\hat{l}+\hat{k}_1)\hat{e}^*_2(\hat{l}+\hat{k}_{12})}{(l+k_1)^2(l+k_{12})^2}\right] + (1\leftrightarrow 2)\right\}. \end{aligned} \tag{7.83}$$

However, it is easy to perform shifts of the loop momenta of the type $l \to l - k_{1,2}$ forcing contribution due to Tr_2 to vanish. This allows us to re-write Eq.(7.80) in a simplified form

$$
\begin{aligned}
\mathcal{M}_{ABJ} &= \int \frac{d^D l}{i(\pi)^{(D/2)}} \mathcal{I}_{ABJ}(k_1, k_2, e_1, e_2, l), \\
\mathcal{I}_{ABJ} &= \frac{-2ie^2}{(4\pi)^{(D/2)}} \mathrm{Tr}\left\{ \Gamma_{\gamma_5} \hat{l}_\epsilon \frac{\hat{l}\hat{e}_1^*(\hat{l}+\hat{k}_1)\hat{e}_2^*(\hat{l}+\hat{k}_{12})}{l^2(l+k_1)^2(l+k_{12})^2} + (1 \leftrightarrow 2) \right\}.
\end{aligned}
\tag{7.84}
$$

Since the integrand $\mathcal{I}_{ABJ}$ is proportional to $\hat{l}_\epsilon$, its cut-constructible part vanishes and its OPP parametrization becomes simple

$$
\begin{aligned}
\mathcal{I}_{ABJ}(l) = & \frac{c_1 l_\epsilon^2}{d_0 d_1 d_{12}} + \frac{c_2 l_\epsilon^2}{d_0 d_2 d_{12}} + \frac{b_1 l_\epsilon^2}{d_0 d_1} + \frac{b_2 l_\epsilon^2}{d_0 d_2} \\
& + \frac{b_3 l_\epsilon^2}{d_1 d_{12}} + \frac{b_4 l_\epsilon^2}{d_2 d_{12}} + \frac{b_5 l_\epsilon^2}{d_0 d_{12}}.
\end{aligned}
\tag{7.85}
$$

We use $d_0 = l^2$, $d_1 = (l+k_1)^2$, $d_2 = (l+k_2)^2$ and $d_{12} = (l+k_{12})^2$ in Eq.(7.85). Although we work under the assumption that electrons are massless, we note that all the manipulations we did up to now remain valid also for massive electrons[4]. In the massless electron case some of the bubble integrals are scaleless and therefore vanish, but the residues of the corresponding integrands do not vanish and are, in fact, mass-independent. We will show below that $c_1 = c_2$ and $b_{i=1,..5} = 0$.

In the case of closed fermion loops the Dirac algebra has to be performed in six dimensions with five-dimensional loop momentum $l = l_{(4)} + l_\epsilon$, where using the notation $(l_0, l_1, l_2, l_3, l_4, l_5)$ we have that $l_\epsilon = (0,0,0,0,\mu,0)$ and $l_\epsilon^2 = -\mu^2$. As explained in Section 6.2, for six-dimensional Dirac matrices we use the simple representation

$$
\begin{aligned}
&\Gamma^0 = \begin{pmatrix} \gamma^0 & \mathbf{0} \\ \mathbf{0} & \gamma^0 \end{pmatrix}, \quad \Gamma^{i=1,2,3} = \begin{pmatrix} \gamma^i & \mathbf{0} \\ \mathbf{0} & \gamma^i \end{pmatrix}, \\
&\Gamma^4 = \begin{pmatrix} \mathbf{0} & \gamma_5 \\ -\gamma_5 & \mathbf{0} \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \mathbf{0} & i\gamma_5 \\ i\gamma_5 & \mathbf{0} \end{pmatrix}, \quad \Gamma_{\gamma_5} = \begin{pmatrix} \gamma_5 & \mathbf{0} \\ \mathbf{0} & \gamma_5 \end{pmatrix}.
\end{aligned}
\tag{7.86}
$$

Note that for our choice of l, Γ^5 never appears in Eq.(7.84). Finally, we choose a special reference frame where

$$
\begin{aligned}
k_{12} = (m,0,0,0,0,0), &\quad k_{1,2} = \left(\frac{m}{2}, \pm\frac{m}{2}, 0,0,0,0\right), \\
e_{1,2}^* = \frac{1}{\sqrt{2}}(0,0,1,\pm i,0,0), &\quad l_\perp = \alpha e_1^* + \beta e_2^*.
\end{aligned}
\tag{7.87}
$$

With this choice of the polarization vectors, it is clear that $e_i^* k_j = 0\,,\ (i = 1,2)$. The coefficients $c_1, c_2, b_1, \ldots b_5$ can be obtained by evaluating triple cuts and double cuts on both side of equation (7.85).

We begin by considering the triple cut specified by the condition $d_0 = d_1 = d_{12} = 0$. Decomposing the loop momentum on the cut as

$$
l_{c_1}^\mu = x_1 k_1^\mu + x_2 k_2^\mu + \tilde{l}^\mu, \quad \tilde{l}^\mu = l_\perp^\mu + l_\epsilon^\mu, \tag{7.88}
$$

[4] In the massive electron case, the divergence of the axial current involves a canonical term $\partial_\mu J_5^\mu = 2m\bar{\psi}\gamma_5\psi$, which should be treated separately.

we find that $x_1 = -1$, $x_2 = 0$ and $\tilde{l}^2 = l_\perp^2 + l_\epsilon^2 = 0$. Taking the d_0, d_1, d_{12} residue of the left hand side of Eq. (7.85) we obtain

$$\begin{aligned}
&\mathrm{Res}\,(\mathcal{I}_{ABJ})\,|_{d_0=d_1=d_{12}=0} = \\
&\frac{-2ie^2}{(4\pi)^{(D/2)}}\mathrm{Tr}\left\{\Gamma_{\gamma_5}\hat{l}_\epsilon(\hat{l}_\perp + \hat{l}_\epsilon - \hat{k}_1)\hat{e}_1^*(\hat{l}_\perp + \hat{l}_\epsilon)\hat{e}_2^*(\hat{l}_\perp + \hat{l}_\epsilon + \hat{k}_2)\right\}.
\end{aligned} \tag{7.89}$$

It follows from Eq.(7.89) that the triple cut residue is the fourth-order polynomial in l_ϵ. However, it is easy to argue that only limited number of terms can contribute to the trace. Indeed, for our choice of the loop momentum, $\hat{l}_\epsilon$ is proportional to Γ^4 in Eq.(7.86) while all other terms in Eq.(7.89) are linear combinations of $\Gamma^{0,1,2,3}$. Since the former is block off-diagonal while the latter are block-diagonal, terms with odd number of l_ϵ's do not contribute to the trace. In addition, for the trace in Eq.(7.89) to be non-zero, at least four γ matrices are needed in addition to Γ_{γ_5}.

Since $\hat{l}_\epsilon$ anticommutes with all other matrices of the trace, the term with four $\hat{l}_\epsilon$ vanishes. We conclude that the only term that contributes to the trace is quadratic in $\hat{l}_\epsilon$. Finally, because $l_\perp$ can be written as a linear combination of e_1^* and e_2^*, only terms that contain two $\hat{l}_\epsilon$ and no $l_\perp$ terms give non-vanishing contributions. Taking all this into account, we arrive at a simple expression for the trace and the residue

$$\mathrm{Res}\,(\mathcal{I}_{ABJ})\,|_{d_0=d_1=d_{12}=0} = \frac{2ie^2}{(4\pi)^{(D/2)}}\mathrm{Tr}\left\{\Gamma_{\gamma_5}\hat{l}_\epsilon(\hat{k}_1)\hat{e}_1^*(\hat{l}_\epsilon)\hat{e}_2^*(\hat{k}_2)\right\}. \tag{7.90}$$

Since the residue of the right hand side in Eq.(7.85) is $c_1 l_\epsilon^2$ we derive the value of the c_1 coefficient

$$c_1 = -\frac{2^{D/2+1}e^2}{(4\pi)^{\frac{D}{2}}}\varepsilon^{\mu\nu\lambda\rho}e_{1\mu}^* k_{1\nu} e_{2\lambda}^* k_{2\rho}. \tag{7.91}$$

Finally, because of the $1 \leftrightarrow 2$ symmetry, we find $c_2 = c_1$.

We next proceed to the double cuts. Apart from obvious changes in the physical and transverse spaces, the only new feature is that on the right hand side of Eq.(7.85) we get the double cut contribution also from the triple pole terms. We illustrate the calculation of the double-pole terms taking $d_0 = d_1 = 0$, as an example. Although this is a double cut, the reference momentum is light-like $k_1^2 = 0$, so the parametrization is subtle. We parametrize the loop momentum on the double-cut as

$$l_{b_1}^\mu = x_1 k_1^\mu + x_2 k_2^\mu + \tilde{l}^\mu \tag{7.92}$$

and use $l^2 = 0, l\cdot k_1 = 0$ to find $x_2 = 0$ and $\tilde{l}^2 = 0$, while x_1 is unconstrained. We compute the d_0, d_1 residue of the left-hand side of Eq.(7.85) using the expression in Eq.(7.84). We obtain

$$\begin{aligned}
\mathrm{Res}\,(\mathcal{I}_{ABJ})\,|_{d_0=d_1=0} &= \frac{-2ie^2}{(4\pi)^{\frac{D}{2}}}\frac{1}{(l+k_{12})^2}\mathrm{Tr}\Big\{\Gamma_{\gamma_5}\hat{l}_\epsilon(\hat{l}_\perp + \hat{l}_\epsilon + x_1\hat{k}_1) \\
&\times \hat{e}_1^*(\hat{l}_\perp + (1+x_1)\hat{k}_1 + \hat{l}_\epsilon)\hat{e}_2^*(\hat{l}_\perp + \hat{l}_\epsilon + x_1\hat{k}_1 + \hat{k}_{12})\Big\}.
\end{aligned} \tag{7.93}$$

Similar to the triple-cut case considered earlier, only terms quadratic in l_ϵ contribute. The non-vanishing terms are proportional to k_2 and, after some algebra, we find that all terms proportional to x_1 cancel and the result is simply expressed through the c_1 coefficient in Eq.(7.91)

$$\mathrm{Res}\,(\mathcal{I}_{ABJ})\,\Big|_{(d_0=d_1=0)} = c_1 \frac{l_\epsilon^2}{(l+k_{12})^2}. \tag{7.94}$$

Since the (d_0, d_1)-residue in Eq.(7.85) is $c_1 l_\epsilon^2/d_{12}+b_1$, we find $b_1 = 0$. A similar calculation proves that other bubble coefficients also vanish. To obtain the final result, we need the value of the triangle integral

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\epsilon^2}{d_0 d_1 d_2} = -\frac{1}{2} + \mathcal{O}(D-4), \tag{7.95}$$

and the value of the coefficient c_1 from Eq.(7.91). Adding the two triangle contributions and setting $D = 4$, we obtain the anomalous amplitude $\mathcal{M}_{ABJ}$ shown in Eq.(7.79).

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